APMA 0365 Practice Exam

The bridgework exam will be out of 100 points.

Problem 1 (20 points)

State the maximum principle for $u_t = Du_{xx}$ with $(x,t) \in [0,L] \times [0,T]$ for given $D, L, T > 0$.

Problem 2 (20 points)

Prove the following statement. Suppose that $u(x,t)$ is a solution of

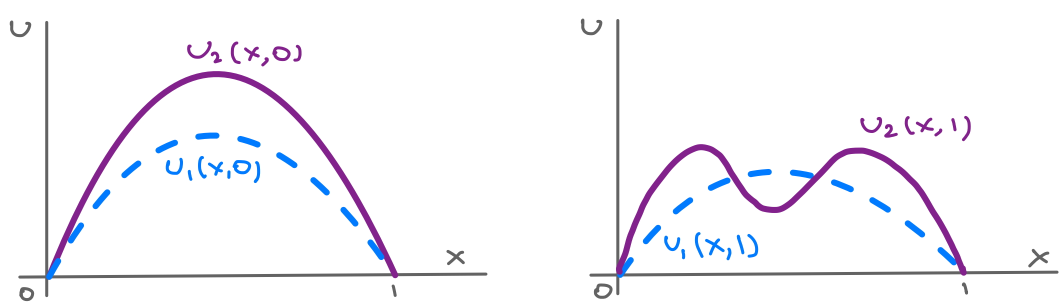
$$\begin{aligned} &u_t = u_{xx} && 0 \leq x \leq 1,\ t > 0 \\ &u(0,t) = 0 = u(1,t) && t \geq 0 \end{aligned}$$

with $u(x,0) \geq 0$ for $0 \leq x \leq 1$, then $u(x,t) \geq 0$ for all $0 \leq x \leq 1$ and $t \geq 0$.

Problem 3 (20 points)

Short answers: you do not need to justify your answers.

1. True or False: Each solution to $u_t + u_x = 0$ is of the form $u(x,t) = f(x-t)$.
2. True or False: Assume $u(x,t)$ is a population of bacteria that move via diffusion in the interval $(0,L)$, then the conditions $u(0,t) = 0 = u(L,t)$ for all $t \geq 0$ guarantee that bacteria cannot enter or leave the interval $(0,L)$.
3. True or False: Assume $u(x,t)$ satisfies $u_t + cu_x = 0$ for all (x,t), then $u(x,t)$ also satisfies $u_{tt} = c^2 u_{xx}$ for all (x,t).
4. True or False: Assume $f \in \mathcal{C}^2$ and $g \in \mathcal{C}^1$ are given, then $u_{tt} = u_{xx}$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ has a unique solution for $x \in \mathbb{R}$ and $t \geq 0$.
5. True or False: Suppose $u_1(x,t)$ and $u_2(x,t)$ both satisfy $u_t = u_{xx}$ for $0 \leq x \leq 1$ with $u(0,t) = 0 = u(1,t)$ for all $t \geq 0$ and that the graphs of these functions at $t = 0$ look as in the left panel below, then it is possible that the graphs of these functions at $t = 1$ look as shown in the right panel below:



Problem 4 (20 points)

Short answers: you do not need to justify your answers.

1. True or False: If f is Schwartz class, then $\mathcal{F}(f_x(x)) = \mathrm{i}k\hat{f}(k)$.
2. Write down the strong maximum principle for harmonic functions.
3. True or False: If f, g are Schwartz, then the convolution $f \star g$ is Schwartz.
4. True or False: If $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, then so does $\sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx)$.
5. Write down the Poisson formula.

Problem 5 (15 points)

Assume $f(x)$ is Schwartz class and let $a \in \mathbb{R}$. Prove that $\mathcal{F}(\mathrm{e}^{\mathrm{i}ax} f(x)) = \hat{f}(k-a)$.

Problem 6 (20 points)

Use the Fourier transform to find a solution $u(x,t)$ of $u_t = u_{xx} - u$ for $x \in \mathbb{R}$ and $t > 0$ with initial condition $u(x,0) = f(x)$ in Schwartz class. You can use the function $G(x,t) := [F^{-1}(\mathrm{e}^{-k^2 t})](x)$ without calculating this inverse Fourier transform explicitly.

Problem 7 (15 points)

Prove that $\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} \sin(nx) = -\sum_{n=1}^{\infty} n^2 \mathrm{e}^{-n^2 t} \sin(nx)$ for $0 \leq x \leq \pi$ and each $t > 0$. You can refer to theorems we proved in class that guarantee that we can switch the order of differentiation and summation – you need to verify the conditions of the assumptions of these theorems.

Problem 8 (15 points)

Let D be open and bounded. Assume that $u \in C^2(D) \cap C^0(\overline{D})$ satisfies $\Delta u \geq 0$ in D. Prove that $u(\vec{x}) \leq \max_{\vec{y} \in \partial D} u(\vec{y})$ for all $\vec{x} \in \overline{D}$.

Problem 9 (15 points)

Let $D = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 4\}$ be the disk of radius $a = 2$ centered at the origin and assume that $u \in C^2(D) \cap C^0(\overline{D})$ is harmonic in D with $u(2\cos\varphi, 2\sin\varphi) = \cos(\varphi) - 10$ for $0 \leq \varphi \leq 2\pi$.

1. Find the maximum value of u in $\overline{D}$.
2. Calculate the value of u at the origin.
