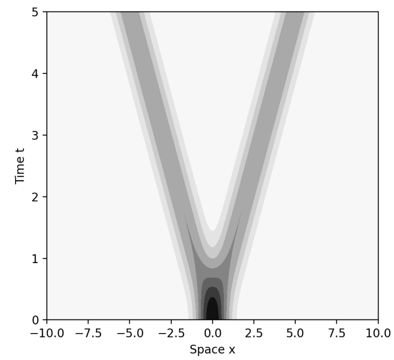
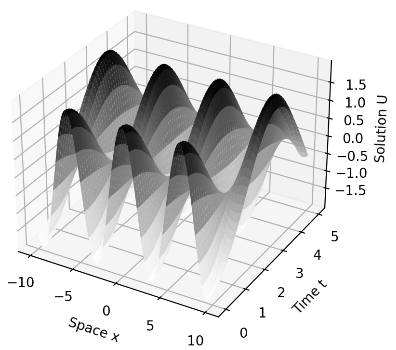
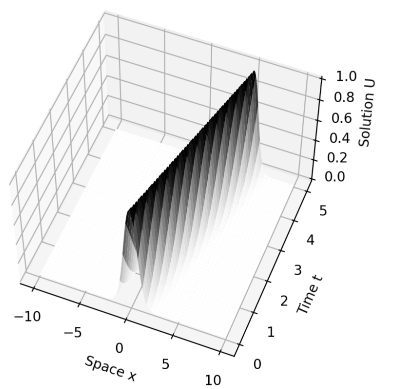
#1.1 Visualizing solutions using contour and surface graphs (5 points)

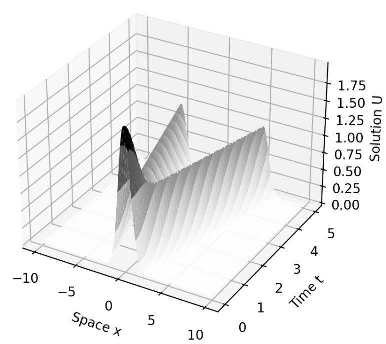
Match the contour graph



with the correct surface plot below







Argue why your chosen graph is the correct match, and why the other graphs cannot match.

#1.2 The superposition principle (5 points)

1. Prove the superposition principle for the wave equation: Assume $c \in \mathbb{R}$ is given. If $u_1(x,t)$ and $u_2(x,t)$ are solutions of $u_{tt} = c^2 u_{xx}$, then so is $u(x,t) = au_1(x,t) + bu_2(x,t)$ for each fixed choice of constants $a, b \in \mathbb{R}$.
2. Next, we will generalize this principle. Assume that $a_{ij}(y)$, $b_j(y)$, and $c(y)$ are continuous functions defined for $y \in \mathbb{R}^n$ with $i, j = 1, \ldots, n$. Consider the linear second-order PDE

   $$\sum_{i,j=1}^{n} a_{ij}(y) u_{y_i y_j} + \sum_{j=1}^{n} b_j(y) u_{y_j} + c(y) u = 0 \tag{1}$$

   for $y \in \mathbb{R}^n$. Prove the following: If $u_1(y)$ and $u_2(y)$ are solutions of (1), then so is $u(y) = d_1 u_1(y) + d_2 u_2(y)$ for each fixed choice of constants $d_1, d_2 \in \mathbb{R}$.
3. Show that case 1. is a direct consequence of the general case 2.

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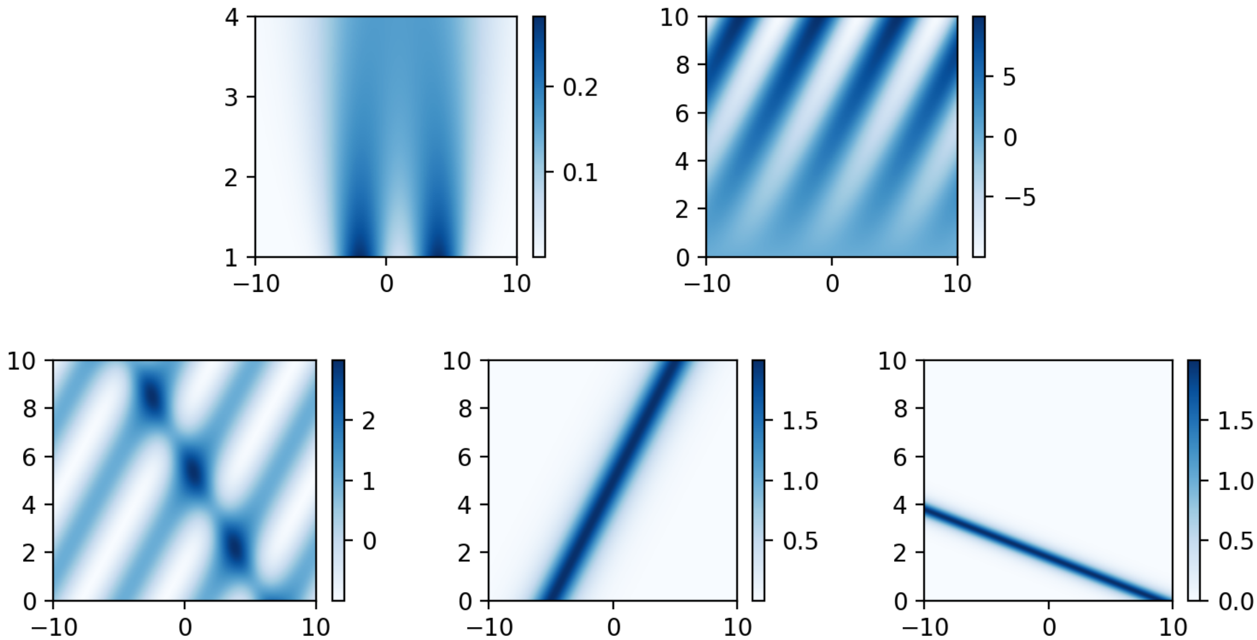
#1.3 Contour plots of solutions to the advection and wave equation (5 points)

In the table provided below, use "X"s to match the PDEs with the contour plots (space horizontally and time vertically) of their solutions. Each row can potentially contain between zero and five "X"s. Justify all your matches and non-matches! Be careful to look at all the information provided, including the ranges (indicated by the values on the axes) over which x and t vary.

	upper left	upper right	lower left	lower center	lower right
$u_{tt} = u_{xx}$					
$u_t + u_x = 0$					



#1.4 The comparison principle for the heat equation (8 points)

Prove the comparison principle for the heat equation $u_t = u_{xx}$:

Lemma: If u_1 and u_2 are two solutions of $u_t = u_{xx}$, and if $u_1(x,t) \leq u_2(x,t)$ when $t = 0$, when $x = 0$, and when $x = \ell$, then $u_1(x,t) \leq u_2(x,t)$ for all (x,t) with $t \geq 0$ and $0 \leq x \leq \ell$.

Hints: Let $u(x,t) := u_1(x,t) - u_2(x,t)$.

- What PDE does $u = u_1 - u_2$ satisfy?
- What can you say about $u(x,t)$ when $t = 0$, when $x = 0$, and when $x = \ell$?
- Which theorem from class about the heat equation could you now use?

Continued on the next page

#1.5 Heat equation (4 points)

We consider

$$\text{(2)} \qquad \begin{cases} u_t = u_{xx} & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0 = u(1,t) & t > 0 \\ u(x,0) = x^2(1-x) & 0 \le x \le 1. \end{cases}$$

Separately for each of the following cases, argue whether my claim is possible or definitely false.

1. I claim that I found a solution $u(x,t)$ of (2) such that $u(x,5) = 2x\sin(\pi x)$.
2. I claim that I found a solution $u(x,t)$ of (2) so that $u(0.2,5) = 0$.

Hint: This problem does not involve any lengthy computations ...

#1.6 Heat equation (4 points)

We consider

$$\text{(3)} \qquad \begin{cases} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0 = u(\pi,t) & t > 0. \end{cases}$$

We proved in class that $v(x,t) = \mathrm{e}^{-t}\sin(x)$ is a solution of (3). Prove or disprove the following statement:

Claim: Let $u(x,t)$ be a solution of (3) with $0 \le u(x,0) \le \sin(x)$, then $\max_{0\le x\le\pi} |u(x,t)| \le \mathrm{e}^{-t}$ as $t \to \infty$.

Hint: You can use any results you proved earlier in the semester ...

#1.7 Heat equation (6 points)

We consider

$$\text{(4)} \qquad \begin{cases} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0 = u(\pi,t) & t > 0 \\ u(x,0) = f(x) & 0 \le x \le \pi, \end{cases}$$

where $f(x)$ is given by

$$f(x) := \begin{cases} 1 & 0 \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \le \pi. \end{cases}$$

1. Draw the graph of $f(x)$ and prove that $f \in L^2(0,\pi)$.
2. Find the formal series solution $u(x,t)$ for (4) with the given initial condition (you need to evaluate all integrals in the formula for the coefficients).
3. Argue why the formal series solution $u(x,t)$ you derived in the previous step is the solution of (4); in particular, in what sense is the initial condition satisfied?
4. Does the series $u(x,t)$ converge uniformly on $[0,\pi]$ for each fixed $t > 0$? Why or why not?
5. Does the series at $t = 0$ converges uniformly on $[0,\pi]$ to the initial condition? Argue why or why not.

Continued on the next page

#1.8 Fourier coefficients for smooth functions (6 points)

We revisit the complete orthogonal set $(\sin(nx))_{n\geq 1}$ of eigenfunctions in $L^2(0,\pi)$. We had stated in class that the series $\sum_{n=1}^{N} c_n \sin(nx)$ converges in L^2 to a given $f \in L^2(0,\pi)$ as $N \to \infty$ provided we choose the coefficients c_n according to

$$c_n := \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x, \qquad n \geq 1. \tag{5}$$

1. Prove the following: Assume $f \in \mathcal{C}^4([0,\pi])$ is four times continuously differentiable and satisfies $f(y) = f''(y) = 0$ for $y = 0, \pi$, then there is a constant $K > 0$ so that $|c_n| \leq K/n^4$ for all $n \geq 1$.
2. Argue why this implies that $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nt)\sin(nx)$ can be differentiated twice term-by-term in t (and also in x).

Hint: For the first part, note that

$$\sin(nx) = -\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}(\cos(nx)) \qquad \text{and} \qquad \int_a^b u(x)v'(x)\,\mathrm{d}x = \Big(u(x)v(x)\Big)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

#1.9 Inhomogeneous boundary conditions for the heat and wave equation (4 points)

We consider

$$\begin{cases} u_t = u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) = T_0 & t > 0 \\ u(L,t) = T_1 & t > 0 \end{cases} \tag{6}$$

for given constants $T_0, T_1 \in \mathbb{R}$. We want to reduce this equation to the case of homogeneous Dirichlet conditions where we set $u(y,t) = 0$ at $y = 0, L$.

1. Find the simplest possible C^2-function $w(x)$ defined on $[0,L]$ that satisfies $w(0) = T_0$ and $w(L) = T_1$.
2. Define $u(x,t) = v(x,t) + w(x)$. Assume that $u(x,t)$ satisfies (6), what is the system of PDE and boundary conditions that $v(x,t)$ will satisfy? The goal is to find a linear system for $v(x,t)$: if your system is not linear, revisit the first step!
3. If $u(x,0) = f(x)$ for $0 \leq x \leq L$, what initial condition would $v(x,t)$ satisfy?
4. Would this approach also work for the wave equation? Argue why or why not.

#1.10 True – False questions (4 points)

1. Assume that $v_n(x)$ are a complete set of eigenfunctions of a Sturm–Liouville problem. Fill in "if", "then", "provided", "implies", or " " for the two PLACEHOLDERs below to make the statement true:
   PLACEHOLDER $\sum_{n=1}^{N} a_n v_n(x) \to f(x)$ uniformly as $N \to \infty$ PLACEHOLDER $f(x)$ is C^2 and satisfies the boundary conditions in the Sturm–Liouville problem.
2. Check off the correct box below. The statement

   $$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} a_n \cos nx = -\sum_{n=1}^{\infty} n a_n \sin nx \quad \text{holds} \quad \begin{cases} \square \text{ always} \\ \square \text{ sometimes depending on } a_n \\ \square \text{ never.} \end{cases}$$

3. Is the following statement correct? "Since $\lambda_n = -n^2$ and $v_n(x) = \cos nx$ satisfy $v_{xx} = \lambda v$ for $0 < x < \pi$ and $v_x(0) = 0 = v_x(\pi)$ for $n \geq 2$, each $f \in L^2(0,\pi)$ can be written as $f \overset{L^2}{=} \sum_{n=2}^{\infty} a_n \cos nx$ for appropriate coefficients a_n". Justify your answer briefly. (Read this very carefully!)

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#1.11 Laplace equation (4 points)

Suppose that $u(x,y)$ is a harmonic function in the disk $D = \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 < 4\}$. In polar coordinates, $(x,y) = (r\cos\varphi, r\sin\varphi)$, assume that $u = 3\sin(2\varphi)+1$ when $r=2$. Without finding the solution, answer the following questions.

1. Find the maximum value of u in $\overline{D}$.
2. Calculate the value of u at the origin.

#1.12 Laplace equation with right-hand side (4 points)

Let $D = \{(x,y) \colon x^2+y^2 < 1\}$ and consider

$$\begin{cases} \Delta u = 4 & (x,y) \in D \\ u(x,y) = 0 & (x,y) \in \partial D. \end{cases} \tag{7}$$

1. Find a solution $u(x,y)$ of (7). *Hint:* This requires virtually no calculations. Perhaps start with the Laplace equation $u_{xx} = 2$ with $-1 < x < 1$ and $u(\pm 1) = 0$.
2. Prove that solutions to (7) are unique.
3. Determine the minimum and maximum of $u(x,y)$ and find all points $(x,y) \in \overline{D}$ at which the minimum and maximum of $u(x,y)$ are achieved.
4. Reconcile your results with the maximum principle we proved in class.

Rubric: Make sure you justify your arguments.

#1.13 Subharmonic functions (6 points)

Let D be open, bounded, and connected. Assume that $u \in C^2(D) \cap C^0(\overline{D})$ satisfies $\Delta u \geq 0$ in D. Prove or disprove the following statements:

1. There is an $\vec{x}_M \in \partial D$ so that $u(\vec{x}) \leq u(\vec{x}_M)$ for all $\vec{x} \in \overline{D}$.
2. There is an $\vec{x}_m \in \partial D$ so that $u(\vec{x}) \geq u(\vec{x}_m)$ for all $\vec{x} \in \overline{D}$.

Rubric: Make sure you justify your arguments.

#1.14 Strong maximum principle and connectedness (4 points)

Let D be open and bounded, but not connected. Prove or disprove the following statement:

1. If $u \in C^2(D) \cap C^0(\overline{D})$ satisfies $\Delta u = 0$ in D and is not a constant function, then there are $\vec{x}_m, \vec{x}_M \in \partial D$ so that $u(\vec{x}_m) < u(\vec{x}) < u(\vec{x}_M)$ for all $\vec{x} \in D$.

Rubric: Make sure you justify your arguments.

#1.15 Fourier transform (2 points)

Assume f is Schwartz. Prove that

$$\mathcal{F}^{-1}\left(\hat{f}(k)\mathrm{e}^{\mathrm{i}ak}\right)(x) = f(x+a), \qquad x \in \mathbb{R}$$

for each $a \in \mathbb{R}$.

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#1.16 The linear Korteweg–de Vries equation (7 points)

Use the Fourier transform to derive a formula for the solution of $u_t + u_{xxx} = 0$ with $u(x,0) = f(x)$ for $f(x)$ in Schwartz class using the definition $G(x,t) := \mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}k^3 t})$ (you do not need to compute this inverse Fourier transform). You do not need to verify that the functions that appear in your analysis are Schwartz class.

#1.17 Conservation laws (10 points)

Consider the nonlinear conservation law

$$\begin{aligned} u_t + u u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \end{aligned} \tag{8}$$

belonging to the flux $\phi(u) = \frac{1}{2}u^2$.

(a) Use the method of characteristics to write down an explicit solution $u(x,t)$ that is defined for $x \leq 0$ and $x > t$.

(b) In the remaining region $0 < x \leq t$, we write $z := x/t$ and seek solutions in the form $u(x,t) = g(x/t)$. Use the PDE to find an explicit expression for $g(x/t)$ that satisfies $g'(z) \neq 0$.

(c) Use the previous two parts to write down an explicit solution $u(x,t)$ for $x \in \mathbb{R}$ and $t > 0$.

(d) Provide a sketch of the contour plot of the solution you found in the previous parts.

(e) Next, use the Rankine–Hugoniot condition to write down a second solution $u(x,t)$ of (8). Provide a sketch of the contour plot of this new solution.

(f) Compare the two solutions!
