1 Advection equation: Characteristic lines

We will derive the characterization $u(xt) = f(x - ct)$ of solutions $u(x,t)$ of the advection equation $u_t + cu_x = 0$ in yet another way. Throughout, we fix $c \in \mathbb{R}$ and assume that $u(x,t)$ is continuously differentiable in (x,t) and satisfies $u_t(x,t) + cu_x(x,t) = 0$ for all $x, t \in \mathbb{R}$.

Our goal is to characterize curves $(x,t) = (x(t),t)$ along which $u(x(t),t)$ stays constant, that is, does not depend on t. In other words, we want to find the contour curves in the (x,t) plane that correspond to the same value of the function $u(x,t)$.

Let's assume we found a C^1-function $x(t)$ so that $u(x(t),t)$ does not depend on t.

Task 1:

1. Define $V(t) := u(x(t),t)$ so that, by our assumption, $V(t)$ does not depend on t. Since $V(t)$ does not depend on t, what can you say about $\frac{\mathrm{d}V}{\mathrm{d}t}(t)$?
2. Next, use the chain rule to express $\frac{\mathrm{d}V}{\mathrm{d}t}(t)$ in terms of derivatives of $u(x,t)$.
3. In the next step, use the assumption that $u(x,t)$ satisfies the advection equation $u_t + cu_x = 0$ to derive an ODE that $x(t)$ has to satisfy to guarantee that $u(x(t),t)$ does indeed not depend on t.
4. Solve the ODE you just derived for $x(t)$ for an arbitrary initial condition x_0.
5. In summary, for each x_0, you found a function $x(t)$ so that $x(0) = x_0$ and such that $u(x(t),t)$ depends only on x_0 but not on t. Use this information to prove that $u(x,t)$ is a function of $x - ct$ only.

Task 2: Our next goal is to generalize this approach to a larger class of advection equations. Assume that $c(x)$ is a given C^1 function of x. Follow the steps given above to derive an ODE for $x(t)$ so that $u(x(t),t)$ does not depend on t whenever $x(t)$ is a solution of this ODE, where we assume that $u(x,t)$ satisfies the more general advection equation $u_t + c(x)u_x = 0$.

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2 Heat equation: Uniqueness via the maximum principle

Task 3: Prove the following uniqueness result for the heat equation $u_t = u_{xx}$ using the maximum principle.

Lemma: Let $u(x,t)$ be a solution of $u_t = u_{xx}$ with $u(x,0) = 0$ for $0 \le x \le L$ and $u(0,t) = u(L,t) = 0$ for $t \ge 0$. Prove that then necessarily $u(x,t) = 0$ for all (x,t) with $t \ge 0$ and $0 \le x \le L$.

Task 4: Next, let ϕ, f, g, u_0 be given functions. Apply your lemma to show that solutions to the system

$$\begin{array}{rcll} u_t &=& u_{xx} + \phi(x,t) & 0 < x < L,\ t \ge 0 \\ u(x,0) &=& u_0(x) & 0 < x < L \\ u(0,t) &=& f(t) & t \ge 0 \\ u(L,t) &=& g(t) & t \ge 0 \end{array}$$

are unique (if they exist). *Hint:* Assume that u_1 and u_2 are two solutions, what system does $u := u_1 - u_2$ satisfy?

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3 Laplace equation in a square

Consider the Laplace equation

$$(1) \qquad \begin{cases} u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\ u(0,y) = 0 = u(\pi,y) & 0 \le y \le \pi \\ u(x,0) = 0, \quad u(x,\pi) = f(x) & 0 \le x \le \pi \end{cases}$$

with Dirichlet conditions except at the top edge $y = \pi$, where $f \in L^2(0,\pi)$ is given.

Task 5: Show that $u(x,y) = X(x)Y(y)$ satisfies (1) iff there is a constant λ so that $X(x)$ and $Y(y)$ satisfy

$$X_{xx}(x) = \lambda X(x), \qquad Y_{yy}(y) = -\lambda Y(y).$$

Task 6: Derive the boundary conditions that $X(x)$ and $Y(y)$ need to satisfy at the left, right, and bottom side, where u satisfies Dirichlet conditions $u = 0$.

Task 7: Find all values of λ and all functions $X(x)$ that satisfy the linear ODE and the boundary conditions you just derived for $X(x)$. You can use any results we derived earlier this semester without re-deriving them, so this task should not require any computations!

Task 8: For each eigenvalue λ_n, find the functions $Y_n(y)$ that satisfy the linear ODE for $Y(y)$ and the Dirichlet boundary conditions at $y = 0$. You can use the following facts without further proof:

The solutions of $Y_{yy} = n^2 Y$ for $n \geq 1$ are of the form $Y_n(y) = a\cosh(ny) + b\sinh(ny)$, where

$$\cosh(y) := \frac{\mathrm{e}^y + \mathrm{e}^{-y}}{2}, \quad \sinh(y) := \frac{\mathrm{e}^y - \mathrm{e}^{-y}}{2}$$

with $\cosh(0) = 1$ and $\sinh(0) = 0$ (we also have $\cosh'(0) = 0$ and $\sinh'(0) = 1$).

Task 9: Show that the formal series solution of the PDE with the given boundary conditions except for the top edge at $y = \pi$ is given by

$$u(x, y) = \sum_{n=1} c_n \sinh(ny)\sin(nx).$$

Task 10: Determine the coefficients c_n so that $u(x, \pi) = f(x)$ in L^2.

Task 11: Let $u_1(x, y)$ be the solution to (1) we just found and assume we also found a solution $u_2(x, y)$ of the different boundary-value problem

$$(2) \qquad \begin{cases} u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\ u(0, y) = 0, \quad u(\pi, y) = g(y) & 0 \leq y \leq \pi \\ u(x, 0) = 0 = u(x, \pi) & 0 \leq x \leq \pi \end{cases}$$

for a given $g \in L^2(0, \pi)$. Can you write down the solution to

$$(3) \qquad \begin{cases} u_{xx} + u_{yy} = 0 & 0 < x, y < \pi \\ u(0, y) = 0, \quad u(\pi, y) = g(y) & 0 \leq y \leq \pi \\ u(x, 0) = 0, \quad u(x, \pi) = f(x) & 0 \leq x \leq \pi \end{cases}$$

assuming we found u_1 and u_2? *Hint:* Try superposition! How would you tackle the problem when we solve the Laplace equation and require that the solution is equal to given functions at each of the four edges of the square domain?

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4 Proofs using the maximum principle

Let $T > 0$ and define $R = (0, \infty) \times (0, T]$ and $\bar{R} = [0, \infty) \times [0, T]$. Assume that $u(x, t)$ is continuous for $(x, t) \in \bar{R}$ and that u is C^1 in t and C^2 in x with $u_t = u_{xx}$ for all $(x, t) \in R$. Finally, we assume that there is an $L > 0$ so that $|u(x, t)| \leq \mathrm{e}^{-x}$ for $(x, t) \in [L, \infty) \times [0, T]$. If $u(0, 0) = 1$, I claim that $u(x, t)$ attains its maximum on the set $\partial R = [0, \infty) \times \{0\} \cup \{0\} \times [0, T]$ and nowhere else.

Task 12: Prove or disprove this claim (with the entire group if in C01 or C02). Here are a few items to consider:

- We want to use the maximum principle, so let's first write it down.
- Draw two figures to visualize (i) your version of the maximum principle and (ii) the setting of my claim. Identify commonalities and differences.
- Make sure that my claim and your version of the maximum principle use different variables for everything. Then match the names of variables in my claim and your theorem in as much as possible – this allows us to see where we have to work to put my claim into the framework of your maximum principle.
- Decide what we need to prove to apply the maximum principle. There will likely be different aspects that we need to verify or reformulate. If so, decide on the order in which you want to check these.
- Prove or disprove the statements you need to check.

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5 Laplace equation on exterior disks

Fix $a > 0$ and consider the exterior $D = \{(x, y)\colon x^2 + y^2 > a^2\}$ of the disk of radius a centered at the origin. Let $h : [0, 2\pi] \to \mathbb{R}$ be continuous with $h(0) = h(2\pi)$. Consider

$$(4) \qquad \begin{cases} \Delta u = 0 & (x, y) \in D \\ u(a\cos\varphi, a\sin\varphi) = h(\varphi) & 0 \le \varphi \le 2\pi. \end{cases}$$

Task 13: Construct the solution $u(r\cos\varphi, r\sin\varphi)$ of (1) that stays bounded as $r \to \infty$.

Hint: Follow the derivation of the Poisson formula and see which modifications are needed.

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6 Uniqueness of solutions to the heat equation

Task 14: Assume that $f(x)$ is Schwartz class. Prove the following: If $v_1(x, t)$ and $v_2(x, t)$ are Schwartz class for each $t > 0$ and satisfy

$$(5) \qquad \begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = f(x) & x \in \mathbb{R}, \end{cases}$$

then necessarily $v_1(x, t) = v_2(x, t)$ for all $x \in \mathbb{R}$ and $t > 0$.

Hint: The following approach is often referred to as the energy method:

- Write down the equation that $w(x, t) = v_2(x, t) - v_1(x, t)$ satisfies.
- Multiply both sides of the resulting PDE by w and integrate in x over $\mathbb{R}$.
- Use that $ww_t = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(w^2)$.
- Use that $\int_{\mathbb{R}} ww_{xx}\,\mathrm{d}x = -\int_{\mathbb{R}}(w_x)^2\,\mathrm{d}x$.
- Show that w must vanish.

7 Properties of solutions to the heat equation

For the next parts, we can use that the solution to (5) is necessarily given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_R \exp\left(-\frac{(x-y)^2}{4t}\right) f(y)\,\mathrm{d}y.$$

Task 15: Show that $\int_{\mathbb{R}} u(x, t)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$ for all $t > 0$. *Hint:* It is easiest to work directly with the PDE and not use the explicit solution.

Task 16: Prove the following two estimates using the solution formula given above:

- $|u(x, t)| \le \sup_{y\in\mathbb{R}} |f(y)|$ for all $x \in \mathbb{R}$ and $t > 0$. *Hint:* Check PSet #1.1 (2).
- $|u(x, t)| \le \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{\mathbb{R}} |f(y)|\,\mathrm{d}y$ for all $x \in \mathbb{R}$ and $t > 0$.

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8 Method of characteristics

Task 17: Consider the linear advection equation

$$(6) \qquad u_t + c(x)u_x = b(x)u, \qquad x \in \mathbb{R}, \quad t > 0,$$

where $c(x)$ and $b(x)$ are given C^1-functions that depend on x. Assume that $u(x, t)$ is a solution, let $x(t)$ be any C^1-function, and define $h(t) := u(x(t), t)$. Use the assumption that $u(x, t)$ satisfies (6) to derive a system of ODEs that $(h, x)(t)$ need to verify. *Hint:* Calculate $\frac{\mathrm{d}h}{\mathrm{d}t}(t)$ and compare with (6).