APMA 1655: Bridgework Pset

- Φ will always denote the cumulative distribution function of the standard normal distribution. That is
$$\Phi(x) = P(N(0,1) \leq x), \quad \text{for all } x.$$

1. Toss a coin with $P(H) = 0.4$ ten times. Let X be the total number of tails. What is the distribution of X?

2. John tosses a fair coin n times. Afterwards, Emily tosses the coin as many times as the number of heads John has obtained. Let X be the number of heads Emily gets. What is the distribution of X? (*Hint: construct suitable binomial trials*)

3. Randomly choose two points from interval $[0, 1]$. What is the probability that the distance between these two points is at least 0.5?

4. Your friend has two coins. One is a regular fair coin with $P(H) = 0.5 = P(T)$. The other is a two-headed coin with $P(H) = 1$. Your friend randomly (that is, equally likely) picks a coin and tosses it twice. Given that both tosses are heads, what is the probability that the fair coin was chosen?

5. Let X be a continuous random variable uniformly distributed on interval $[0, 1]$.
   (a) What is $P(0.2 \leq X \leq 0.5)$?
   (b) Define $Y = \sqrt{X}$. Compute the cumulative distribution function of Y.
   (c) What is the probability density function of Y.
   (d) Let n be an arbitrary natural number. Compute $E[X^n]$ and $\text{Var}[X^n]$.

6. Let X and Y be two independent identically distributed random variables uniformly distributed on $[0, 1]$. Let $Z \doteq \min(X, Y)$. What is the probability density function for Z? What is $E[Z]$?

7. Let X and Y be two independent exponentially distributed random variables, with X exponential with rate λ and Y exponential with rate μ. Let $Z \doteq \min(X, Y)$. What is the probability density function for Z?

8. Let X be uniformly distributed on $[0, 1]$. Let $Y = X^2$. What is the cumulative distribution function of Y? What is the probability density function of Y?

9. Let (X, Y) be a continuous random vector with joint probability density function

$$f(x, y) = \begin{cases} cx & \text{if } 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases},$$

   where c is some positive constant.
   (a) What should be the value of c?
   (b) What is the (marginal) probability density function for X?
   (c) Fix an arbitrary $0 < x < 1$. What is the conditional probability density function of Y, given $X = x$?
   (d) What is $E[Y|X = x]$?

10. Let X and Y be independent normal random variables with $X \sim N(0, 3^2)$ and $Y \sim N(1, 4^2)$. Compute

$$P(X + Y \geq 6).$$

Express your result in terms of Φ, the CDF of the standard normal.

11. Suppose you first randomly pick a number from the exponential distribution with rate 1, say X [i.e., X is exponentially distributed with rate 1]. Once you have X, you randomly pick a nonnegative integer Y from the Poisson distribution with parameter equal to X. Compute $P(Y = 0)$.

12. A random number (say) X is chosen uniformly from $(0, 1)$. Once it is chosen, John will toss a coin with probability of heads equal to X, and Emily will toss a coin with probability of heads equal to $1 - X$. Assume that they toss independently. Find the probability that both John and Emily toss a heads.

13. Let $X_1, X_2, \ldots, X_n$ be iid (independent and identically distributed) random variables such that

$$P(X_i = 0) = P(X_i = 1) = P(X_i = 2) = \frac{1}{3}.$$

Define X to be the product of these random variables. That is, $X = X_1 \cdot X_2 \cdot \cdots \cdot X_n$.

(a) What is $P(X = 0)$?

(b) What is $E[X]$?

(c) What is $\text{Var}[X]$?

14. Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables with common probability density function

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Compute $E[X_1]$ and $\text{Var}[X_1]$.

(b) In the case where $n = 200$, use central limit theorem (CLT) to approximate the probability

$$P(X_1 + X_2 + \cdots + X_{200} \geq 130).$$

Express your final answer in terms of Φ.

15. Let $X_1, X_2, \ldots, X_n$ be iid samples from the exponential distribution with mean θ. That is, the population probability density function takes the form

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad x \geq 0.$$

(a) Find the maximum likelihood estimate (MLE) for θ.

(b) Is this MLE consistent?

(c) It this MLE unbiased?

(d) What is the mean square error of this MLE?

(e) When n is large, what is the approximate distribution of this MLE?

16. A coin is tossed n times and let X denote the number of heads. Consider the following two estimates for the probability of heads $p = P(H)$:

$$\hat{p}_1 = \frac{X}{n}, \qquad \hat{p}_2 = \frac{X+1}{n+2}.$$

(a) Which one of these two estimates are unbiased?

(b) Are both estimates consistent?

(c) Is the mean square error of $\hat{p}_1$ always less than that of $\hat{p}_2$?

17. Let $X_1, X_2, \ldots, X_n$ be iid sample from $N(0, \sigma^2)$ with unknown σ^2. Find the maximum likelihood estimate of σ^2. Is this estimate unbiased? Is it consistent?

18. Suppose A and B are two independent events. Show that A^c and B are also independent.

19. Let X and Y be two random variables with $\mathrm{Var}[Y] > 0$. Find the constant β that minimizes the variance

$$\mathrm{Var}[X + \beta Y].$$

20. For a random variable X with $E[X] = \mu$, find the constant a that minimizes the *squared loss*

$$E[(X - a)^2].$$

A different version of this problem: Show that the *squared loss*

$$E[(X - a)^2] = \mathrm{Var}[X] + (a - E[X])^2.$$

What value of a minimizes the squared loss?

21. Let X be a standard normal random variable. Assume that given $X = x$, Y is normally distributed as $N(x, 1)$. Find $\mathrm{Cov}(X, Y)$.

22. An investor has \$1 million captital and can choose to invest any portion of it. Suppose that y is the amount that invested. Then with probability p the amount invested will double, and with probability $1 - p$ the amount invested will be lost. Let X be the total wealth in the end. Assuming $p > 0.5$, determine the value of y that maximizes the expected utility

$$E[\ln(X)].$$

23. Suppose U is has uniform distribution $U(0, 1)$. Let $X = \frac{1}{U+1}$.

(a) Find the probability that $X > \frac{3}{4}$

(b) Find the cumulative distribution function $F(x)$ and the probability density function $f(x)$ for X. Express your functions as piecewise functions defined on $\mathbb{R}$.

(c) Find $E(X)$.

24. Let X be a geometric random variable with probability of success (on a single trial) equal to $\frac{1}{4}$, what is the probability that X equals an odd integer?

25. A single cell will die with probability p or split into two with probability $1-p$, producing a the second generation of cells. Each cell in the second generation (if there are any) will die or split into two with the same probabilities as the initial cell.
(a) Find the probability distribution function for the number of cells in the third generation.
(b) What is the expected size of the kth generation, where each cell dies or reproduces with the same probabilities as the initial cell?

26. (a)Let X_1 and X_2 be two (discrete) random variables that are independent and identical geometric random variables with parameter p. Give a closed formula for the probability:

$$p(X_1 = i \,|X_1 + X_2 = n)$$

(b) Suppose a bus arrives at a station such that the time between arrivals is exponentially distributed with rate $\frac{1}{\lambda}$. To get home, you decide to wait for the bus for some number of minutes t. If the bus has arrived before t minutes, you take the bus home which takes time B. If the bus has not arrived after t minutes, you walk home which takes time W.

(i) What is the expected total time from getting to the bus stop until getting home?

(ii) Supose $W < \frac{1}{\lambda} + B$ at what value of t is the expected wait time minimized?

(iii) Suppose $W > \frac{1}{\lambda} + B$ at what value of t is the expected wait time minimized?

27. Show that for random variables X and Y

$$V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$$

28. Let A, B, C be events of a common probability space.
(a) Prove the following equation holds:

$$P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$$

(b) Give an example that shows the following does **not** generally hold:

$$P(A|B \cup C) = P(A|B) + P(A|C) - P(A|B \cap C)$$

29. Suppose that Y_1 and Y_2 have joint density function given by

$$f(y_1, y_2) = \begin{cases} c(1-y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid density function.
(b) Find the marginal densities for Y_1 and Y_2.
(c) Find the conditional density for Y_1 given $Y_2 = y_2$.
(d) Compute the probability $p(Y_1 \le .5|Y_2 = .75)$.
(e) Compute the expectations $E[Y_1]$, $E[Y_2]$.
(f) Compute the expectation $E[Y_1|Y_2 = y_2]$.

30. Let X be a random variable and let a, b be real numbers. Prove that $Var[aX+b] = a^2 Var[X]$

31. Let X be a geometric random variable with parameter p. Let n be a given positive integer. Find the conditional probability mass function of X given $X > n$ and $E[X|X > n]$.

32. Let (X, Y) be a continuous random vector with joint probability density function

$$f(x,y) = \begin{cases} 2 & \text{if } x \geq 0, y \geq 0, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

For any $y \in (0, 1)$, compute $E[X \mid Y = y]$.

33. Let (X, Y) be a random vector whose joint probability density function is given by

$$f(x,y) = \begin{cases} x+y & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the (marginal) probability density function of X?
   (b) What is the conditional probability density function of Y given $X = 0.5$?
   (c) Compute the conditional probability $P(Y \leq 0.5 \mid X = 0.5)$.

34. Randomly and independently pick two points X and Y from the interval $[0, 1]$. That is, X and Y are independent random variables uniformly distributed on $[0, 1]$. Compute the probability $P(X + Y \leq 0.5)$.

35. Five people are sitting at a table in a restaurant. Two of them order coffee and the other three order tea. The waiter forgot who ordered what and puts the drinks in a random order for the five persons. Determine the probability that each person gets the correct drink.

36. A bag contains four balls. One is blue, one is white and two are red. Someone draws two balls at random from the bag. He looks at them and tells you that there is a red ball among the two balls drawn out. What is the probability the the other ball drawn out is also red?

37. On the island of liars each inhabitant lies with probability 2/3. You overhear an inhabitant making a statement. Next you ask another inhabitant whether the inhabitant you overheard spoke truthfully. Use Bayes' rule to find the probability that the inhabitant you overheard indeed spoke truthfully given that the other inhabitant says so.

38. Bill and Mark play a series of games until one of the players has won two games more than the other player. Any game is won by Bill with probability p and by Mark with probability $q = 1 - p$. The results of the games are independent of each other. What is the probability that Bill will be the winner of the match? (*Hint:* Use law of total probability – consider all possible outcomes from the first two games)

39. Let A, B and C be arbitrary events with $P(C) > 0$. Assume that A and B are disjoint, that is, $A \cap B = \emptyset$. Show that

   (a) $P(A^c \mid C) = 1 - P(A \mid C)$;

(b) $P(A \cup B \,|\, C) = P(A \,|\, C) + P(B \,|\, C)$.

40. Consider a random variable X with

$$P(X = 0) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{3}, \quad P(X = -1) = \frac{1}{6}$$

(a) Draw the probability mass function and cumulative distribution function of X.

(b) What are $E[X]$ and $\mathrm{Var}[X]$?

41. Suppose X is a random variable with $E[X] = 1$ and $\mathrm{Var}[X] = 4$. Determine the following quantities:

$$E[X^2], \quad E[(X+1)^2], \quad \mathrm{Var}[2X], \quad \mathrm{Var}[-2X+1].$$

42. A fair die is rolled twice. Let X denote the larger number rolled (if two numbers are the same, then the larger number is defined to be the common number rolled). Find the probability mass function of X.

43. Coupons in cereal boxes are numbered 1 to 5. And a set of one of each is required for a prize. With one coupon per cereal box, how many boxes on the average are required to make a complete set? *Hint: A geometric random variable with parameter p has expected value* $1/p$.

44. Let X be an arbitrary random variable. Denote $\mu = E[X]$ and $\sigma = \mathrm{Std}[X]$. Assume that $\sigma \neq 0$. Define

$$Z \doteq \frac{X - \mu}{\sigma}.$$

Show that $E[Z] = 0$ and $\mathrm{Var}[Z] = 1$. **Remark:** This is said to be the *standardization* of the random variable X.

45. Randomly select a point from a disc of radius 1. Denote by X its distance to the center. Find the cumulative distribution function, the probability density function, and the expected value of X.

46. Consider the interval $(0, 1)$ and a given fixed point $s \in (0, 1)$. Choosing at random a point in $(0, 1)$ divides this interval into two subintervals. What is the expected length of the subinterval covering the prefixed point s?

Remark: Your answer should be bigger than 1/2. This is indeed a manifestation of selection bias. The longer of the two subintervals is more likely to be the one that covers the point s. Compare this with selecting either subinterval equally likely, in which case the average length will be 1/2.

47. Let Q be an arbitrarily fixed point on the circumference of a circle with radius one. Choose at random a point P on the circumference of the circle and let the random variable X be the length of the *line* segment between P and Q. What is the expected value of X? (*Hint:* Say O is the center of the circle. Let Θ be the angle between PO and QO. What should be the distribution of Θ? Then write the length of PQ as a function of Θ)

48. Let X be a nonnegative random variable, that is, $P(X \geq 0) = 1$. Let $f(x)$ be the probability density function of X. Let $Y = X^2$. Determine the probability density function for Y.

49. The joint density function of the continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} cxy & \text{for } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Identify the constant c and compute $P(X \leq 0.5)$ and $E[Y]$.

50. Emily has a fair coin and Pete has a coin with $P(H) = p$. Emily tosses her coin twice. Now Pete tosses his coin as many times as the number of heads Emily gets. Let X and Y be the numbers of heads Emily and John get, respectively.

    (a) Find the joint probability mass function of (X, Y). You might want to use a table to describe this pmf.

    (b) What is the (marginal) distribution of X?

    (c) What is the (marginal) distribution of Y?

    (d) Without any calculation, do you think $\text{Cov}(X, Y)$ is positive or negative? Why?

    (e) Compute $\text{Cov}(X, Y)$ to verify your answer in (d).

51. The joint density function of the continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} 8xy & \text{for } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

    (a) Without any calculation, can you tell if X and Y are independent?

    (b) Compute $\text{Cov}(X, Y)$.

52. Randomly select two points on the interval $[0, 1]$. What is the probability that the distance between these two points is less than 0.5?

53. Let X and Y be independent identically distributed (iid) geometric random variables with parameter p. Find the distribution of $Z = \min(X, Y)$.

54. Let X and Y be independent standard normal random variables. Let a be an arbitrarily fixed real number. Define

$$Z = aX + Y, \qquad W = aX - Y$$

    (a) What is the distribution of Z? What is the distribution of W? Are Z and W identically distributed?

    (b) Compute $\text{Cov}(Z, W)$.

    (c) For what values of a are Z and W independent?

55. A fair coin is tossed three times. Denote by X the number of heads among the first two tosses, and Y that of the last two tosses. Find the conditional probability mass function of X given $Y = 1$, $E[X|Y = 1]$ and $\text{Var}[X|Y = 1]$.

56. The continuous random variables X and Y satisfy $f(y|x) = 1/x$ for $0 < y < x$ and $f(y|x) = 0$ otherwise. The marginal density function of X is given by $f_X(x) = 2x$ for $0 < x < 1$ and $f_X(x) = 0$ otherwise. What is the joint density $f(x, y)$? What is the conditional density $f(x|y)$? What is $E[X|Y = y]$?

57. Let $\{X_1, X_2, \ldots\}$ be a sequence of iid (independent identically distributed) random variables uniformly distributed on $(0, 1)$. Define for each $n \geq 1$,

$$Y_n = \frac{1}{n}\left(X_1 1_{\{X_2<0.5\}} + X_3 1_{\{X_4<0.5\}} + \cdots + X_{2n-1} 1_{\{X_{2n}<0.5\}}\right).$$

(a) What should be the limit of Y_n as $n \to \infty$?

(b) What should be the approximate distribution of Y_n when n is large?

58. (Going broke playing a favorable game, hard) Here is an example where aggressive bet sizing can lead to bankruptcy, even if the game is really favorable. Consider a game, which you can play repeatedly. In each round, if you bet x amount, then with 49% chance you will double your stake (i.e., net winning amount x) and 51% chance you will lose half of your stake (i.e., net winning amount $-x/2$). Here we always assume that $x > 0$.

(a) Compute your average net winning, if you bet x amount. [Convince yourself that the game is really favorable. Actually, the number you get should be very close to $0.25x$, in other words, roughly a 25% gain on average for each play]

(b) You can play this game repeatedly, as many times as you would like. The outcomes from different rounds can be assumed to be independent. You have an initial wealth $S > 0$ (the specific value of S is not important). Your goal is to get obnoxiously rich by playing this game. Alas, life is short and you want to get rich quick, so you decide to go *all-in* in each round. For example, you will bet S in the first round. If you win the first round (you total wealth is now $2S$), then you will bet $2S$ in the second round; If you lose the first round (you total wealth is now $S/2$), then you will bet $S/2$ in the second round. And so on. After all, this is a game very much in your favor. Nothing can go wrong.

Show that, your total wealth will go to zero if you keep playing this game, with 100% probability.

[*Hint: Denote S_n the wealth after n rounds. Try to write* $\log(S_n/S)$ *as the sum of appropriate iid random variables. Then use SLLN*]

59. The concentration of dissolved metals can be dangerously high in water draining through abandoned mine tunnels. Suppose the trace-metal concentrations in two streams in Montana were investigated. Independent random 1-liter samples of water were obtained near each mine, and the amount of zinc (in micrograms per liter) in each sample was measured. The resulting summary statistics are given in the following table. The samples from these two streams are considered to be independent.

Location	Sample size	Sample mean	Sample standard deviation
Jack Creek	15	993.1	28.5
Cataract Creek	17	968.6	29.8

(a) Give a 95% confidence interval for the concentration of zinc in Jack Creek.

(b) Give a 95% confidence interval for the difference of zinc concentrations in Jack Creek and Cataract Creek.

60. To compare the proportions of people who favor a certain proposal in two large communities, independent random samples were taken from both communities.

	sample size	in favor
Community A	60	48
Community B	100	50

Let p_A and p_B denote the population proportion favoring the proposal in community A and community B, respectively. Give the 95% confidence interval for the difference $p_A - p_B$.

61. Identify the maximum likelihood estimate (MLE) of the unknown parameter in each of the following cases, where $X_1, \ldots, X_n$ are iid samples from

    (a) geometric distribution with unknown probability of success p;

    (b) normal distribution with (known) mean 0 and unknown variance σ^2 [for this case, find the MLE for σ].

62. Let $X_1, X_2, \cdots, X_n$ be iid samples from density

$$f(x) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the maximum likelihood estimate for θ. Explain, using Strong Law of Large Numbers, that this maximum likelihood estimate is consistent.

63. In lectures, all the examples of MLE assume that the samples are iid. This assumption is convenient but not at all essential in the theory of MLE. All we need is the ability to write down the likelihood function.

Consider the following problem. Let $X_1, X_2, \ldots, X_n$ be independent, but not identically distributed, samples. These X_i's are assumed to be normally distributed with

$$X_i \sim N(\theta c_i, 1), \qquad i = 1, 2, \ldots, n,$$

where θ is an unknown parameter and c_i's are some known constants (not all c_i's are zero). We wish to estimate θ.

    (a) Write down the likelihood function $L_\theta(x_1, x_2, \ldots, x_n)$, i.e., the joint probability density function of $(X_1, \ldots, X_n)$.

    (b) Find the MLE for θ, that is, the maximizer of $L_\theta(X_1, X_2, \ldots, X_n)$.

64. In an attempt to verify the claim that the absentee rate in Friday morning class is higher than that in Monday to Thursday morning classes, independent random samples of students were taken. They yield the following results.

	sample size	number of absentee
Friday	50	14
Monday-Thursday	150	18

(a) Clearly identify the null hypothesis and alternative hypothesis.

(b) Calculate the p-value of this test.

(c) Is the test result significant at the 5% significance level?

65. (Two-sided test and confidence interval) Let $X_1, X_2, \ldots, X_n$ be iid samples from $N(\theta, 1)$. The population parameter θ is unknown. Our goal is to estimate or test θ. Consider the two sided test

$$H_0: \quad \theta = \theta_0, \qquad H_a: \quad \theta \neq \theta_0$$

where θ_0 is given. Assume the significance level is α. Prove that the null hypothesis H_0 will be rejected at significance level α if and only if the confidence interval of confidence level $(1-\alpha)$ does not contain θ_0.

Remark: Here we have assumed the population variance is known ($\sigma^2 = 1$). This is not essential since one can replace standard deviation with standard error. Secondly, the assumption of normal distribution is also not essential when sample size is large, since $\bar{X}_n$ will be approximately normal by CLT.

66. An advertisement at a Casino slot machine claims that a person has a 1/3 chance to win at each play. Suppose we have reason to believe that the real chance is actually lower. We employ a one-sided hypothesis test as follows: We plan to play the slot machine three times, and we will reject the advertisement if we lose every time.

(a) What is the Type I error for this test?

(b) Suppose we think the real chance to win is 1/5. Compute the Type II error and power under this assumption.

67. An experiment was conducted to observe the effect of temperature (in Fahrenheit) on the potency of an antibiotic. Below is the data observed from the experiment:

Potency Reading (y)	37	35	30	24	23	20	17	13
temperature (x)	30°	40°	50°	60°	70°	80°	90°	100°

The R^2 for this data set is a whopping 0.98! We used MATLAB to fit a linear regression model:

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

and the numerical output is as follows:

$$\hat{\beta}_0 = 47.238, \qquad \hat{\beta}_1 = -0.344,$$

$$\text{SE}(\hat{\beta}_0) = 1.354, \qquad \text{SE}(\hat{\beta}_1) = 0.020, \qquad \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -0.025.$$

The regression line is also shown in the above picture.

(a) Construct a 95% confidence interval for the average change in potency if one increases the temperature by 10°.

(b) Is there statistically significant evidence that the potency of the antibiotic decreases when the temperature increases? Assume the significance level is $\alpha = 0.01$.

(c) Suppose we wish to estimate the average potency reading at temperature $x^* = 45°$. Find your estimate and compute its standard error. Do you have faith in this estimate?

(d) Suppose we wish to estimate the average potency reading at temperature $x^* = -100°$. Find your estimate and compute its standard error. Do you have faith in this estimate?

68. Conside the simple regression model for a data set (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Let

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

be the resulting regression line. Show that $(\hat{\beta}_0, \hat{\beta}_1)$ is also the maximum likelihood estimate, if we further assume that in the model

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i,$$

where the errors ε_i are iid (independent identically distributed) normally distributed as $N(0, \sigma^2)$.

69. Suppose that $X_1, \ldots, X_n$ are iid (independent identically distributed) samples with the common probability density function

$$f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

where $\theta > 0$ is the unknown parameter. Find the MLE (maximum likelihood estimate) of θ.

70. Let $X_1, \ldots, X_n$ be iid Poisson random variables with parameter λ. Consider the following estimate for λ:

$$\hat{\lambda} = \frac{(X_1 + X_2 + \cdots + X_n) + 1}{n + 1}$$

(a) Is this estimate unbiased?

(b) Is this estimate consistent? Which theorem did you use to arrive at your conclusion?

(c) Compute the Mean Square Error (MSE) of the estimate $\hat{\lambda}$.

71. Suppose that $X_1, X_2, \ldots, X_n$ are iid random variables with probability density function $f(x)$. Consider the following hypothesis testing problem

$$H_0: \; f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}, \qquad H_a: \; f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that we reject the null hypothesis if and only if every one of $X_1, X_2, \ldots, X_n$ is larger than (say) $x^* = 0.1$.

(a) What is the Type-I error of this test?

(b) What are the power and Type-II error of this test?

72. Suppose that $X_1, X_2, \ldots, X_n$ are iid samples with distribution $N(\theta, 10^2)$. Let $\overline{X}$ denote the sample mean, that is,

$$\overline{X} \doteq \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

Consider the following test

$$H_0: \ \theta = 0, \qquad H_a: \ \theta > 0$$

Let $n = 100$. Suppose that we reject the null hypothesis if and only if the sample mean $\overline{X}$ exceeds 2. Please solve the following problems and express the answers in terms of Φ.

(a) What is the Type I error of this test?

(b) What are the power and Type II error, if the true value of θ equals 1?

73. A study measured the growth of body weight of leghorn male chicks after birth:

- x: the age of the chick, measured in days;
- Y: the weight of the chick, measured in the unit of grams.

The study used 25 chicks (the oldest among these chicks was around 30 days old). The age and weight of each chick were measured. A linear regression model

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

was fit and the numerical output was as follows:

$$\hat{\beta}_0 = 100, \qquad \hat{\beta}_1 = 2.5,$$

$$\mathrm{SE}(\hat{\beta}_0) = 10, \qquad \mathrm{SE}(\hat{\beta}_1) = 0.3, \qquad \mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -2.$$

(a) Find the 95% confidence interval of a chick's average-weight-gain-per-day.

(b) Suppose that we wish to estimate the average weight of a chick of 100 days old. Find your estimate and the 95% confidence interval. Do you have faith in this estimate?