APMA 0355 Bridge Work Practice

Problem 1

State Grönwall's lemma.

Problem 2

1. Consider $\dot{x} = Ax$ where $A \in \mathbb{R}^{n \times n}$, and assume that $x_1(t)$ and $x_2(t)$ are solutions. For each choice of constants c_1, c_2 prove that $x(t) := c_1 x_1(t) + c_2 x_2(t)$ is also a solution.
2. Consider $\dot{x} = Ax + g(t)$ where $A \in \mathbb{R}^{n \times n}$ and $g : \mathbb{R} \to \mathbb{R}^n$ is a continuous. Prove that if $x_1(t)$ and $x_2(t)$ are solutions, then $y(t) := x_1(t) - x_2(t)$ satisfies $\dot{y} = Ay$.

Problem 3

Consider $\dot{u} = f(u)$ where $u \in \mathbb{R}$. We assume that there is an $L > 0$ so that $|f(u) - f(v)| \leq L|u - v|$ for all $u, v \in \mathbb{R}$. Prove the following: if $u_1(t)$ and $u_2(t)$ are solutions of $\dot{u} = f(u)$, then $|u_1(t) - u_2(t)| \leq |u_1(0) - u_2(0)| \mathrm{e}^{Lt}$ for all $t \geq 0$.

Problem 4

For each of the following statements, state whether it is True or False. You do not need to justify your answers.

1. Our existence and uniqueness theorem shows that the differential equation $\dot{x} = \sqrt{|x|} \cos(t)$ has a unique solution $x(t)$ with $x(1) = 0$.
2. If $\dot{x} = f(x)$ has two distinct solutions that both satisfy $x(0) = x_0$, then $f(x)$ cannot be continuous in x for all x.
3. If $f(x,t)$ and $\frac{\mathrm{d}f}{\mathrm{d}x}(x,t)$ are continuous in (x,t) for all x, t, then the differential equation $\dot{x} = f(x,t)$ with the initial condition $x(t_0) = x_0$ has a unique solution defined for t in some interval (a,b) with $a < t_0 < b$.
4. If $\dot{x} = f(x)$ has a unique solution for each given initial condition $x(t_0) = x_0$, then $f(x)$ is continuously differentiable.
