APMA 0355: Problem Set #2

#2.3 (8 points) Statements and implications

Respond to the following prompts and provide arguments and justification of your answers:

(1) Write down the negation of the statement "My cat is grey and Anna's dog is white".

(2) Write down your interpretation of the statement "Anna or Paul will go to the shop": if the statement is true, who will go to the shop? Recall: "or" in mathematics is not exclusive (so it is not the same as "either ... or ..."). Finally, write down the negation of the statement.

(3) Is the sentence "If I am at the Brown bookstore, I have books around me" an implication? Is it true? What can you say when I am not in the Brown bookstore: am I around books or not?

(4) Write down the converse of the statement "If I am at the Brown bookstore, I have books around me". Is the converse implication true?

(5) Let $A =$"I am at the Brown bookstore" and $B =$"I have books around me". Write down the statement $A \Leftrightarrow B$ as an English sentence. Explain what the statement $A \Leftrightarrow B$ means in this context. Finally, discuss whether the statement $A \Leftrightarrow B$ is true, that is whether A and B are equivalent.

APMA 0355: Problem Set #3

#3.1 (8 points) Superposition principle

Assume that $b : \mathbb{R} \to \mathbb{R}$ is a continuous function and consider the differential equation

$$\dot{x} = b(t)x. \tag{1}$$

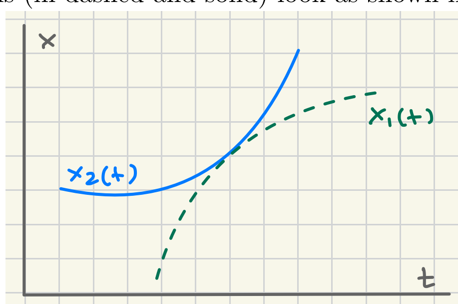
Claim: The sum of two solutions of (**??**) is also a solution.

1. Formulate the claim as a lemma in the form of an implication.
2. Prove your lemma.
3. Can we extend the lemma to equations of the form $\dot{x} = b(t)x + c(t)$, where $b(t)$ and $c(t)$ are continuous?

This result is often referred to as the superposition principle.

#3.2 (4 points) Uniqueness of solutions

Consider the differential equations $\dot{x} = f(x,t)$ where we know that $f(x,t)$ is continuous. Suppose you found two solutions $x_1(t)$ and $x_2(t)$ whose graphs (in dashed and solid) look as shown in the figure below:



What can you say about $f(x,t)$? (*Hint:* Use the contrapositive of the existence and uniqueness theorem).

#3.3 (6 points) Existence and uniqueness

Consider the following scalar ODEs:

1. $\dot{x} = x^2(|t| + x)$
2. $\dot{x} = \frac{1}{x+1}$
3. $\dot{x} = \sqrt{|x|}$

What can you conclude, and not conclude, about existence and uniqueness of solutions to each of the ODEs above with the initial condition $x(t_0) = x_0$ for arbitrary $x_0, t_0 \in \mathbb{R}$? Write complete concluding sentences and justify your answers.

APMA 0355: Problem Set #4

#4.2 (3 points) Uniqueness

You and one of your friends are both solving the differential equation

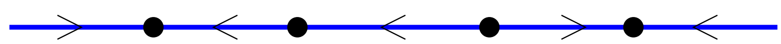
$$\frac{\mathrm{d}u}{\mathrm{d}x} = f(u, x).$$

You obtain the solution $u(x) = x^2$, and your friend gets $u(x) = -x^4$. Could you both be right? Please discuss this question – you can refer to any results we covered in class. If there are multiple cases, please go through all of them.

#4.3 (6 points) Equilibria

For each of the prompts below, find a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the differential equation $\dot{u} = f(u)$ has the stated properties or, if there is no such function, explain in detail (with a proof or a precise argument) why not. If possible, give an explicit expression of the function; otherwise, sketch its graph.

1. Every real number is a rest state.
2. Every integer is an equilibrium, and there are no other equilibria besides those.
3. There are exactly two equilibria, and both of them are stable.
4. There are no equilibria.
5. The phase diagram looks as shown in the picture below:



APMA 0355: Problem Set #7

#7.2 (5 points) Maximal existence

Consider the scalar differential equation $\dot{x} = f(x)$ where we assume that $f(0) = 0$ and that there is a constant $L > 0$ so that $f(x)$ is Lipschitz continuous in x with Lipschitz constant L for all $x \in \mathbb{R}$. Given any $x_0 \in \mathbb{R}$, prove that the solution $x(t)$ of the ODE with $x(0) = x_0$ exists for all $t \geq 0$. You can use without proof that a maximal solution in the sense of Theorem 1 exists. (*Hint: Write down what you know. Then write down what you want to prove based on Theorem 1(iii). Additional hints for this problem are on the next page.*)

#7.3 (5 points) Linear algebra review

1. For which values of λ does the determinant of the matrix $\begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}$ vanish?
2. Draw two nonzero vectors in $\mathbb{R}^2$ that are linearly dependent and, separately, two nonzero vectors that are linearly independent.
3. For which values of a do the vectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} a \\ 1 \end{pmatrix}$ form a basis?
4. For which values of a does the system $Ax = b$ with $A = \begin{pmatrix} 2 & a \\ -1 & 1 \end{pmatrix}$ have a unique solution regardless of what $b \in \mathbb{R}^2$ is?
5. Without solving, argue why the system $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$ must have a non-zero solution $\begin{pmatrix} u \\ v \end{pmatrix}$. Find this solution.

Here is one possible approach to #7.2:

1. Write down what Lipschitz continuity of $f(x)$ with constant L means in formulas.
2. Use $f(0) = 0$ in the expression you wrote down in step 1.
3. Assume that $x(t)$ is the maximal solution on $0 \leq t < T$ for some finite T. What do you know about $|x(t)|$ as t approaches T?
4. Work towards a contradiction that T is finite. Can you use Gronwall's lemma?

APMA 0355: Problem Set #8

#8.1 (5 points) Linear ODE 1

Find the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} x, \qquad x(0) = x_0$$

using matrix exponentials. Provide all steps of your calculations of eigenvalues and eigenvectors. You can leave the final answer as a product of matrices.

#8.2 (5 points) Linear ODE 2

Find the solution $x(t)$ of the first-order system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{pmatrix} -5 & 6 \\ -3 & 4 \end{pmatrix} x \qquad x(0) = x_0$$

using matrix exponentials. Provide all steps of your calculations of eigenvalues and eigenvectors. You can leave the final answer as a product of matrices.

APMA 0355: Problem Set #9

#9.2 (8 points)

Consider the system $\dot{x} = Ax$ of differential equations where $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$.

1. **Superposition principle:** Assume that $x_1(t)$ and $x_2(t)$ are solutions. For any choice of constants $c_1, c_2 \in \mathbb{R}$, prove that $x(t) = c_1 x_1(t) + c_2 x_2(t)$ is also a solution.
2. **Uniqueness of solutions:** Pick any $x_0 \in \mathbb{R}^n$. Assume that $x_1(t)$ and $x_2(t)$ are solutions for $t \geq 0$ that both satisfy the initial condition $x(0) = x_0$. Prove that $x_1(t) = x_2(t)$ for all $t \geq 0$. *Hint:* You can use that for each matrix $A \in \mathbb{R}^{n \times n}$ there is a number L so that $|Ax| \leq L|x|$ for all $x \in \mathbb{R}^n$. The smallest L with this property is called the matrix norm of A and is often denoted by $\|A\|$.

APMA 0355: Problem Set #10

#10.1 (5 points)]

Consider the differential equation $\dot{x} = Ax + b$, where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, and A is a real $n \times n$ matrix. We assume that A is invertible.

1. Find the equilibrium x_* of this system.
2. For any solution $x(t)$, let $y(t) := x(t) - x_*$ denote the displacement of the $x(t)$ from the equilibrium. Derive the differential equation that $y(t)$ satisfies.
3. Use the previous part to write down an explicit formula for the solution $x(t)$ of $\dot{x} = Ax + b$ with $x(0) = x_0$.
4. What conditions do you need to impose to ensure that $x(t) \to x_*$ as $t \to \infty$ regardless of the value x_0 of the initial condition?
