APMA 0355: Worksheet #2

1 Statements and implications

Consider the following sentences:

A: I am in a zoo
B: I am amongst animals
C: Prove that 2 is a prime number
D: My parrot is green and Tina's bird is red
E: Taylor Swift is great

Task 6: Respond to the following prompts and provide arguments and justification of your answers, where A-E will always refer to the sentences above:

(1) Which of the sentences A-E are statements?
(2) Write down the statement $A \Rightarrow B$ as an English sentence. Is this implication true?
(3) Write down the converse of $A \Rightarrow B$ as a formula and as an English sentence. Is the converse true?
(4) What does $A \Leftrightarrow B$ mean in words? Are A and B equivalent?
(5) Write down the contraposition of the statement $A \Rightarrow B$ as an English sentence.
(6) Negate D.

Task 7: Next, we focus on two arbitrary statements X and Y. Respond to the following prompts:

(1) Negate the following statement "X is true or Y is true". (Recall that "M is true or N is true" means that M is true, N is true, or both are true – it is different from "Either M or N are true" which means that one of them is true but not both.)
(2) Negate the following statement "X is true and Y is true".
(3) Negate the following statement "X is true or Y is false".

APMA 0355: Worksheet #3

2 Existence and uniqueness of solution to scalar ODEs

Consider the following scalar ODEs

1. $\dot{x} = ax(1 - x/K)$ for constants a, K
2. $\dot{x} = \mathrm{e}^x - \sin(t)x^2 + g(x)$ where $g(x) = 1$ for $x \geq 0$ and $g(x) = -1$ for $x < 0$
3. $\dot{x} = |t|x^2$.

Task 1: What can you conclude, and not conclude, about existence and uniqueness of solutions to each of the ODEs above when we supplement them with the initial condition $x(t_0) = x_0$ for fixed values of $x_0, t_0 \in \mathbb{R}$? Write complete concluding sentences and justify your answers.

3 Preparing for the proof of the uniqueness theorem

Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that there is a constant $L > 0$ so that $\max_{x \in \mathbb{R}} |f'(x)| \leq L$.

Task 2: Prove that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Check your answers with a TA.

Hint: Which result from calculus relates derivatives and integrals?

4 Stationary solutions

Assume that $f(x)$ is continuously differentiable for all x and consider the scalar ODE $\dot{x} = f(x)$. Suppose that there is an $x_* \in \mathbb{R}$ so that $f(x_*) = 0$.

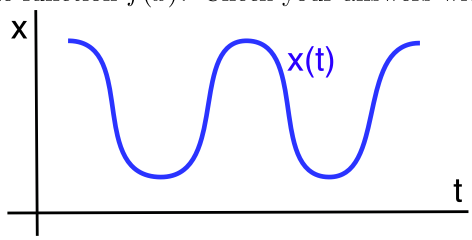
Task 3: Show that the constant function $x(t) = x_*$ for all t is a solution of $\dot{x} = f(x)$.

Solutions of this type are referred to as equilibria (singular: equilibrium), rest states, or stationary solutions because they do not depend on time.

5 Oscillations in scalar ODEs

Many biological and physical processes exhibit oscillations: examples are our sleep cycle, weather and climate seasons, and swinging pendula. We will explore if these phenomena can be modeled by scalar first-order ODEs, where the right-hand side does not depend on time (these ODEs are called autonomous). The precise question is the following:

Task 4: Can the function $x(t)$ graphed below be a solution of a scalar first-order ODE of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ for an appropriate continuously differentiable function $f(x)$? Check your answers with a TA.



Hint: Add the slopes of the graph of $x(t)$ to the figure and compare with the previous task.

6 Trapping regions

Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t),$$

where we assume that $f(x,t)$ and $\frac{\mathrm{d}f}{\mathrm{d}x}(x,t)$ are both continuous for all (x,t) and where $f(x,t)$ satisfies $f(0,t) > 0$ for all $t \geq 0$.

Task 5: Fix $x_0 > 0$ and argue whether the solution $x(t)$ with $x(0) = x_0$ can ever reach the value $x = 0$ at some finite positive value of t or not. Write down a statement of your finding and provide a rigorous proof of it. Check your answers with a TA.

APMA 0355: Worksheet #5

7 Lipschitz continuity

Recall the following fact that you proved in Worksheet #3: Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that there is a constant $L > 0$ so that $\max_{x\in\mathbb{R}} |f'(x)| \leq L$, then $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$.

Task 3: Show that there is a constant L so that $f(x) = \sin(x)$ satisfies $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. What value would you choose for L?

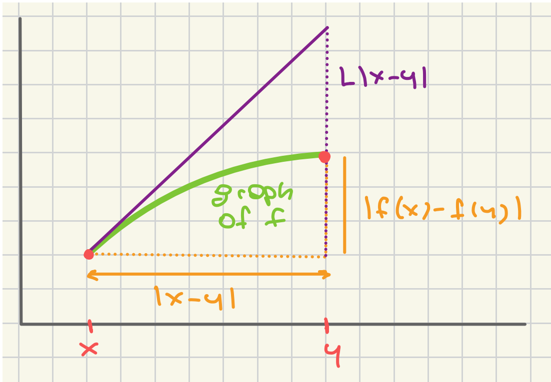
Next, we give functions $f(x)$ that have the property we listed above a name:

Definition: We say that a function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if there is a constant L so that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$.

We care about Lipschitz continuous functions for the following reason: We will prove later that if $f(x)$ is Lipschitz continuous, then solutions of $\dot{x} = f(x)$ are unique!

Task 4: Find conditions that guarantee that a continuously differentiable function is Lipschitz continuous. State a lemma that reflects your findings.

Task 5: Use the graph below to discuss what Lipschitz continuity might mean geometrically. Recall that the function $f(x)$ is fixed and that Lipschitz continuity requires that there is a number L for which $|f(x) - f(y)| \leq L|x - y|$ for all x, y.



Next, we explore whether there are functions (i) that are Lipschitz continuous even though they are not continuously differentiable or (ii) that are continuous but not Lipschitz continuous.

Task 6: Prove the following:

- The function $f_1(x) = |x|$ is Lipschitz continuous with Lipschitz constant $L = 1$.
- The function $f_2(x) = \sqrt{|x|}$ is not Lipschitz continuous. Hint: Draw the graph of $f_2(x)$ and use the geometric intuition we developed above to see where Lipschitz continuity may fail. Alternatively, check where the derivative of $f_2(x)$ may be ill behaved. From there, decide whether a direct or an indirect proof might work best.

APMA 0355: Worksheet #8

8 Matrix exponentials

Task 3: For scalars x and y, we know that $\mathrm{e}^x\mathrm{e}^y = \mathrm{e}^{x+y}$. Here, we explore whether this identity holds for matrices. Consider the matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Compute e^A and e^B using the power-series definition of the matrix exponential. Next recall the expression for e^{A+B} that we derived in class. Is the equation $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B}$ an identity?

Task 4: Let A and B be any two $n \times n$ matrices. Show that, if $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B)$. *Hint:* Show that $x(t) = \mathrm{e}^{At}\mathrm{e}^{Bt}x_0$ is the solution of $\dot{x} = (A + B)x$ with $x(0) = x_0$. Then use without proof that solutions to linear systems are unique to conclude that $\mathrm{e}^{At}\mathrm{e}^{Bt} = \mathrm{e}^{(A+B)t}$ for all t. Be careful when using the product rule and when simplifying expressions: Task 3 shows that you need to use that $AB = BA$ in your calculation, so make sure to note where exactly in your proof you used this assumption. What sanity check could you carry out here given the results from Tasks 3 and 4?

Task 5: Use the results from the previous two tasks to find the solution $x(t)$ of $\dot{x} = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix} x$ with $x(0) = x_0$. Hint: Write $A := \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$ in the form $A = b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = b\mathbb{1} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=:N}$. Do the identity matrix $\mathbb{1}$ and the matrix N commute?

APMA 0355: Worksheet #9

9 Linear systems of ODEs with constant coefficients

Task 1: Find the solution $x(t)$ of $\dot{x} = \begin{pmatrix} 3 & 5 \\ -1 & -1 \end{pmatrix} x$ with $x(0) = x_0$. Sketch the phase portrait.

Task 2: Find the solution $x(t)$ of $\dot{x} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} x$ with $x(0) = x_0$. Sketch the phase portrait. *Hint:* Check Task 5 of Worksheet 8 and proceed similarly.

Task 3: Prove the following: If $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and $b \neq 0$ is an eigenvalue of the matrix $A \in \mathbb{R}^{2\times 2}$ with eigenvector $v = v_1 + iv_2$ for $v_1, v_2 \in \mathbb{R}^2$, then v_1, v_2 are linearly independent. *Hint:* Argue by contradiction: Assume there is a $c \in \mathbb{R}$ with $v_2 = cv_1$ and substitute this into the eigenvalue equation.

Task 4: Summarize and compare the different ways in which we constructed solutions of $\dot{x} = Ax$ where $A \in \mathbb{R}^{n\times n}$. For instance, in Lemma 3, assuming that we found n linearly independent eigenvectors $v_1, \ldots, v_n$ belonging to eigenvalues $\lambda_1, \ldots, \lambda_n$, we found $x(t) = \mathrm{e}^{At} x_0$, $x(t) = V\mathrm{e}^{Dt}V^{-1}$, where V has columns v_j and D is diagonal with entries λ_j on the diagonal, and $x(t) = \sum_{j=1}^{n} c_j \mathrm{e}^{\lambda_j t} v_j$. How are these related? What happens if some of the eigenvalues are not real?

Task 5: If you have time left, start with problem #9.2 on PSet 9 or discuss the final project.