Chapter 1

Essentials of Fractional Calculus

In this chapter we introduce the linear operators of fractional integration and fractional differentiation in the framework of the so-called *fractional calculus*. Our approach is essentially based on an integral formulation of the fractional calculus acting on sufficiently well-behaved functions defined in $\mathbb{R}^+$ or in all of $\mathbb{R}$. Such an integral approach turns out to be the most convenient to be treated with the techniques of Laplace and Fourier transforms, respectively. We thus keep distinct the cases $\mathbb{R}^+$ and $\mathbb{R}$ denoting the corresponding formulations of fractional calculus by *Riemann–Liouville* or *Caputo* and *Liouville–Weyl*, respectively, from the names of their pioneers.

For historical and bibliographical notes we refer the interested reader to the the end of this chapter.

Our mathematical treatment is expected to be accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigour.

Remark : Here, and in all our following treatment, the integrals are intended in the *generalized* Riemann sense, so that any function is required to be *locally* absolutely integrable in $\mathbb{R}^+$. However, we will not bother to give descriptions of sets of admissible functions and will not hesitate, when necessary, to use *formal* expressions with generalized functions (*distributions*), which, as far as possible, will be re-interpreted in the framework of classical functions.

1.1 The fractional integral with support in $\mathbb{R}^+$

Let us consider *causal functions*, namely complex or real valued functions $f(t)$ of a real variable t that are vanishing for $t < 0$.

According to the Riemann–Liouville approach to fractional calculus the notion of fractional integral of order α ($\alpha > 0$) for a causal function $f(t)$, sufficiently well-behaved, is a natural analogue of the well-known formula (usually attributed to Cauchy), but probably due to Dirichlet, which reduces the calculation of the n–fold primitive of a function $f(t)$ to a single integral of convolution type.

In our notation, the Cauchy formula reads for $t > 0$:

$$ {}_0I_t^n\, f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1}\, f(\tau)\, d\tau\,,\ n \in \mathbb{N}\,, \quad (1.1) $$

where $\mathbb{N}$ is the set of positive integers. From this definition we note that $f_n(t)$ vanishes at $t = 0$, jointly with its derivatives of order $1, 2, \ldots, n-1$.

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the *Riemann-Liouville fractional integral* of order $\alpha > 0$:

$$ {}_0I_t^\alpha\, f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, f(\tau)\, d\tau\,,\ t > 0\,,\ \alpha \in \mathbb{R}^+\,, \quad (1.2) $$

where $\mathbb{R}^+$ is the set of positive real numbers. For complementation we define ${}_0I_t^0 := I$ (Identity operator), i.e. we mean ${}_0I_t^0\, f(t) = f(t)$.

Denoting by $\circ$ the composition between operators, we note the *semigroup property*

$$ {}_0I_t^\alpha \circ {}_0I_t^\beta = {}_0I_t^{\alpha+\beta}\,, \quad \alpha\,,\ \beta \ge 0\,, \quad (1.3) $$

which implies the *commutative property* ${}_0I_t^\beta \circ {}_0I_t^\alpha = {}_0I_t^\alpha \circ {}_0I_t^\beta$. We also note the effect of our operators ${}_0I_t^\alpha$ on the power functions

$$ {}_0I_t^\alpha\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad \alpha \ge 0\,, \quad \gamma > -1\,, \quad t > 0\,. \quad (1.4) $$

The properties (1.3) and (1.4) are of course a natural generalization of those known when the order is a positive integer. The proofs are based on the properties of the two Eulerian integrals, i.e. the Gamma and Beta functions, see Appendix A,

$$\Gamma(z) := \int_0^\infty \mathrm{e}^{-u}\, u^{z-1}\, du\,, \quad \mathrm{Re}\,\{z\} > 0\,, \tag{1.5}$$

$$B(p,q) := \int_0^1 (1-u)^{p-1}\, u^{q-1}\, du \;=\; \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,,\ \mathrm{Re}\,\{p\,,\,q\} > 0\,. \tag{1.6}$$

For our purposes it is convenient to introduce the causal function

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\,, \quad \alpha > 0\,, \tag{1.7}$$

where the suffix $+$ is just denoting that the function is vanishing for $t < 0$ (as required by the definition of a causal function). We agree to denote this function as *Gel'fand-Shilov function* of order α to honour the authors who have treated it in their book [Gel'fand and Shilov (1964)]. Being $\alpha > 0$, this function turns out to be *locally* absolutely integrable in $\mathbb{R}^+$.

Let us now recall the notion of *Laplace convolution*, i.e. the convolution integral with two causal functions, which reads in our notation

$$f(t) \,*\, g(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau = g(t) \,*\, f(t)\,.$$

We note from (1.2) and (1.7) that the fractional integral of order $\alpha > 0$ can be considered as the Laplace convolution between $\Phi_\alpha(t)$ and $f(t)$, i.e. ,

$${}_0I_t^\alpha\, f(t) = \Phi_\alpha(t) \,*\, f(t)\,, \quad \alpha > 0\,. \tag{1.8}$$

Furthermore, based on the Eulerian integrals, one proves the *composition rule*

$$\Phi_\alpha(t) \,*\, \Phi_\beta(t) = \Phi_{\alpha+\beta}(t)\,, \quad \alpha\,,\ \beta > 0\,, \tag{1.9}$$

which can be used to re-obtain (1.3) and (1.4).

The Laplace transform for the fractional integral. Let us now introduce the Laplace transform of a generic function $f(t)$, locally absolutely integrable in $\mathbb{R}^+$, by the notation[1]

$$\mathcal{L}\,[f(t);s] := \int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt = \widetilde{f}(s)\,,\ s \in \mathbb{C}\,.$$

By using the sign $\div$ to denote the juxtaposition of the function $f(t)$ with its Laplace transform $\widetilde{f}(s)$, a Laplace transform pair reads

$$f(t)\ \div\ \widetilde{f}(s)\,.$$

Then, for the convolution theorem of the Laplace transforms, see e.g. [Doetsch (1974)], we have the pair

$$f(t)\ *\ g(t)\ \div \widetilde{f}(s)\,\widetilde{g}(s)\,.$$

As a consequence of Eq. (1.8) and of the known Laplace transform pair

$$\Phi_\alpha(t)\ \div\ \frac{1}{s^\alpha}\,, \quad \alpha > 0\,,$$

we note the following formula for the Laplace transform of the fractional integral,

$$ {}_0I_t^\alpha\, f(t) \div \frac{\widetilde{f}(s)}{s^\alpha}\,, \quad \alpha > 0\,, \tag{1.10}$$

which is the straightforward generalization of the corresponding formula for the n-fold repeated integral (1.1) by replacing n with α.

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[1] A sufficient condition of the existence of the Laplace transform is that the original function is of exponential type as $t \to \infty$. This means that some constant a_f exists such that the product $\mathrm{e}^{-a_f t}\,|f(t)|$ is bounded for all t greater than some T. Then $\widetilde{f}(s)$ exists and is analytic in the half plane $\mathcal{R}e\,(s) > a_f$. If $f(t)$ is piecewise differentiable, then the inversion formula

$$f(t) = \mathcal{L}^{-1}\left[\widetilde{f}(s);t\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{\,st}\,\widetilde{f}(s)\,ds\,, \quad \mathcal{R}e\ (s) = \gamma > a_f\,,$$

with $t > 0$, holds true at all points where $f(t)$ is continuous and the (complex) integral in it must be understood in the sense of the Cauchy principal value.

1.2 The fractional derivative with support in $\mathbb{R}^+$

After the notion of fractional integral, that of fractional derivative of order α $(\alpha > 0)$ becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. We note that for this generalization some care is required in the integration, and the theory of generalized functions would be invoked. However, we prefer to follow an approach that, avoiding the use of generalized functions as far is possible, is based on the following observation: the local operator of the standard derivative of order n $(n \in \mathbb{N})$ for a given t, $D_t^n := \dfrac{d^n}{dt^n}$ is just the left inverse (and not the right inverse) of the non-local operator of the n-fold integral ${}_aI_t^n$, having as a starting point any finite $a < t$. In fact, for any well-behaved function $f(t)$ $(t \in \mathbb{R})$, we recognize

$$D_t^n \circ {}_aI_t^n \, f(t) = f(t) \,, \quad t > a \,, \tag{1.11}$$

and

$${}_aI_t^n \circ D_t^n \, f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+) \, \frac{(t-a)^k}{k!} \,, \quad t > a \,. \tag{1.12}$$

As a consequence, taking $a \equiv 0$, we require that ${}_0D_t^\alpha$ be defined as *left-inverse* to ${}_0I_t^\alpha$. For this purpose we first introduce the positive integer

$$m \in \mathbb{N} \quad \text{such that} \quad m-1 < \alpha \le m \,,$$

and then we define the *Riemann-Liouville fractional derivative* of order $\alpha > 0$:

$${}_0D_t^\alpha \, f(t) := D_t^m \circ {}_0I_t^{m-\alpha} \, f(t) \,, \quad \text{with} \quad m-1 < \alpha \le m \,, \tag{1.13}$$

namely

$${}_0D_t^\alpha \, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, \\[2ex] \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{1.13a}$$

For complementation we define ${}_0D_t^0 = I$.

In analogy with the fractional integral, we have agreed to refer to this fractional derivative as the *Riemann-Liouville fractional derivative.*

We easily recognize, using the semigroup property (1.3),

$$ {}_0D_t^{\alpha} \circ {}_0I_t^{\alpha} = D_t^{m} \circ {}_0I_t^{m-\alpha} \circ {}_0I_t^{\alpha} = D_t^{m} \circ {}_0I_t^{m} = I \,. \tag{1.14}$$

Furthermore we obtain

$$ {}_0D_t^{\alpha}\, t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \quad \alpha>0\,, \quad \gamma>-1\,, \quad t>0\,. \tag{1.15}$$

Of course, properties (1.14) and (1.15) are a natural generalization of those known when the order is a positive integer. Since in (1.15) the argument of the Gamma function in the denominator can be negative, we need to consider the analytical continuation of $\Gamma(z)$ in (1.5) into the left half-plane.

Note the remarkable fact that when α is not integer ($\alpha \notin \mathbb{N}$) the fractional derivative ${}_0D_t^{\alpha}\, f(t)$ is not zero for the constant function $f(t) \equiv 1$. In fact, Eq. (1.15) with $\gamma = 0$ gives

$$ {}_0D_t^{\alpha} 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \alpha \geq 0\,, \quad t>0\,, \tag{1.16}$$

which identically vanishes for $\alpha \in \mathbb{N}$, due to the poles of the Gamma function in the points $0, -1, -2, \ldots$.

By interchanging in (1.13) the processes of differentiation and integration we are led to the so-called *Caputo fractional derivative* of order $\alpha > 0$ defined as:

$$ {}_0^*D_t^{\alpha}\, f(t) := {}_0I_t^{m-\alpha} \circ D_t^{m}\, f(t) \quad \text{with} \quad m-1<\alpha\leq m\,, \tag{1.17}$$

namely

$$ {}_0^*D_t^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, & m-1<\alpha<m\,, \\[2ex] \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{1.17a}$$

To distinguish the Caputo derivative from the Riemann-Liouville derivative we decorate it with the additional apex $*$. For non-integer α the definition (1.17) requires the absolute integrability of the derivative of order m. Whenever we use the operator ${}_0^*D_t^\alpha$ we (tacitly) assume that this condition is met.

We easily recognize that in general

$$ {}_0D_t^\alpha\, f(t) := D_t^m \circ {}_0I_t^{m-\alpha}\, f(t) \neq {}_0I_t^{m-\alpha} \circ D_t^m\, f(t) \;=:\; {}_0^*D_t^\alpha\, f(t), \quad (1.18) $$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the exchange of the m-derivative with the integral is legitimate, we have

$$ {}_0^*D_t^\alpha\, f(t) = {}_0D_t^\alpha\, f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\,, \quad (1.19) $$

and therefore, recalling the fractional derivative of the power functions (1.15),

$$ {}_0^*D_t^\alpha\, f(t) = {}_0D_t^\alpha \left[f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right]. \quad (1.20) $$

In particular for $0 < \alpha < 1$ (i.e. $m = 1$) we have

$$ {}_0^*D_t^\alpha\, f(t) = {}_0D_t^\alpha\, f(t) - f(0^+)\, \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = {}_0D_t^\alpha\, \left[f(t) - f(0^+) \right]. $$

From Eq. (1.20) we recognize that the *Caputo* fractional derivative represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. We also note that for its existence all the limiting values,

$$ f^{(k)}(0^+) := \lim_{t\to 0^+} D_t^k f(t),\; k = 0, 1, \ldots m-1\,, $$

are required to be finite. In the special case $f^{(k)}(0^+) \equiv 0$, we recover the identity between the two fractional derivatives.

We now explore the most relevant differences between the two fractional derivatives. We first note from (1.15) that

$$ {}_0D_t^\alpha t^{\alpha-1} \equiv 0\,, \quad \alpha > 0\,, \quad t > 0\,, \quad (1.21) $$

and, in view of (1.20),

$$ {}_0^*D_t^\alpha 1 \equiv 0\,, \quad \alpha > 0\,, \quad (1.22) $$

in contrast with (1.16). More generally, from Eqs. (1.21) and (1.22) we thus recognize the following statements about functions which for $t > 0$ admit the same fractional derivative of order α (in the Riemann-Liouville or Caputo sense), with $m-1 < \alpha \le m$, $m \in \mathbb{N}$,

$$ {}_0D_t^\alpha\, f(t) = {}_0D_t^\alpha\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\alpha-j}\,, \tag{1.23} $$

$$ {}_0^*D_t^\alpha\, f(t) = {}_0^*D_t^\alpha\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,, \tag{1.24} $$

where the coefficients c_j are arbitrary constants. Incidentally, we note that (1.21) provides an instructive example for the fact that ${}_0D_t^\alpha$ is not right-inverse to ${}_0I_t^\alpha$, since for $t > 0$

$$ {}_0I_t^\alpha \circ {}_0D_t^\alpha\, t^{\alpha-1} \equiv 0, \;\; {}_0D_t^\alpha \circ {}_0I_t^\alpha\, t^{\alpha-1} = t^{\alpha-1}\,,\; \alpha > 0\,. \tag{1.25} $$

We observe the different behaviour of the two fractional derivatives in the Riemann-Liouville and Caputo at the end points of the interval $(m-1, m)$, namely, when the order is any positive integer, as it can be noted from their definitions (1.13), (1.17). For $\alpha \to m^-$ both derivatives reduce to D_t^m, as explicitly stated in Eqs. (1.13a), (1.17a), due to the fact that the operator ${}_0I_t^0 = I$ commutes with D_t^m. On the other hand, for $\alpha \to (m-1)^+$ we have:

$$ \begin{cases} {}_0D_t^\alpha f(t) \to D_t^m \circ {}_0I_t^1\, f(t) = D_t^{m-1}\, f(t) = f^{(m-1)}(t)\,, \\ {}_0^*D_t^\alpha f(t) \to {}_0I_t^1 \circ D_t^m\, f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+)\,. \end{cases} \tag{1.26} $$

As a consequence, roughly speaking, we can say that ${}_0D_t^\alpha$ is, with respect to its order α, an operator continuous at any positive integer, whereas ${}_0^*D_t^\alpha$ is an operator only left-continuous.

Furthermore, we observe that the semigroup property of the standard derivatives is not generally valid for both the fractional derivatives when the order is not integer.

The Laplace transform for the fractional derivatives. We point out the major usefulness of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation[2]. In fact, for the Caputo derivative of order α with $m-1<\alpha\le m$, we have

$$\mathcal{L}\left\{ {}_0^*D_t^\alpha\, f(t); s\right\} = s^\alpha\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k}\, f^{(k)}(0^+)\,, \\ f^{(k)}(0^+) := \lim_{t\to 0^+}\ D_t^k f(t)\,. \tag{1.27}$$

The corresponding rule for the Riemann-Liouville derivative of order α is

$$\mathcal{L}\left\{ {}_0D_t^\alpha\, f(t); s\right\} = s^\alpha\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k}\, g^{(k)}(0^+)\,, \\ g^{(k)}(0^+) := \lim_{t\to 0^+}\ D_t^k g(t)\,,\, g(t) := {}_0I_t^{m-\alpha}\, f(t)\,. \tag{1.28}$$

Thus it is more cumbersome to use the rule (1.28) than (1.27). The rule (1.28) requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral. However, when all the limiting values $f^{(k)}(0^+)$ for $k=0,1,\ldots$ are finite and the order is not integer, we can prove that the corresponding $g^{(k)}(0^+)$ vanish so that the formula (1.28) simplifies into

$$\mathcal{L}\left\{{}_0D_t^\alpha\, f(t); s\right\} = s^\alpha\, \widetilde{f}(s)\,, \quad m-1<\alpha<m\,. \tag{1.29}$$

For this proof it is sufficient to apply the Laplace transform to Eq. (1.19), by recalling that

$$\mathcal{L}\left\{t^\alpha; s\right\} = \Gamma(\beta+1)/s^{\alpha+1}\,, \quad \alpha>-1\,, \tag{1.30}$$

and then to compare (1.27) with (1.28).

It may be convenient to simply refer to the Riemann-Liouville derivative and to the Caputo derivative to as R–L and C derivatives, respectively.

We now show how the standard definitions (1.13) and (1.17) for the R–L and C derivatives of order α of a function $f(t)$ ($t\in \mathbb{R}^+$) can

[2] We recall that under suitable conditions the Laplace transform of the m-derivative of $f(t)$ is given by

$$\mathcal{L}\left\{D_t^m\, f(t); s\right\} = s^m\, \widetilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, s^{m-1-k}\,, \quad f^{(k)}(0^+) := \lim_{t\to 0^+}\ D_t^k f(t)\,.$$

be derived, at least *formally*, by the convolution of $\Phi_{-\alpha}(t)$ with $f(t)$, in a sort of analogy with (1.8) for the fractional integral. For this purpose we need to recall from the treatise on generalized functions [Gel'fand and Shilov (1964)] that (with proper interpretation of the quotient as a limit if $t = 0$)

$$\Phi_{-n}(t) := \frac{t_+^{-n-1}}{\Gamma(-n)} = \delta^{(n)}(t)\,, \quad n = 0\,, 1\,, \ldots\,, \tag{1.31}$$

where $\delta^{(n)}(t)$ denotes the generalized derivative of order n of the Dirac delta distribution. Here, we assume that the reader has some minimal knowledge concerning these generalized functions, sufficient for handling classical problems in physics and engineering.

Equation (1.31) provides an interesting (not so well known) representation of $\delta^{(n)}(t)$, which is useful in our following treatment of fractional derivatives. In fact, we note that the derivative of order n of a causal function $f(t)$ can be obtained for $t > 0$ *formally* by the *(generalized) convolution* between Φ_{-n} and f,

$$\frac{d^n}{dt^n}\, f(t) = f^{(n)}(t) = \Phi_{-n}(t) \, * \, f(t) = \int_{0^-}^{t^+} f(\tau)\, \delta^{(n)}(t-\tau)\, d\tau\,, \tag{1.32}$$

based on the well-known property

$$\int_{0^-}^{t^+} f(\tau)\, \delta^{(n)}(\tau - t)\, d\tau = (-1)^n\, f^{(n)}(t)\,, \tag{1.33}$$

where $\delta^{(n)}(t-\tau) = (-1)^n\, \delta^{(n)}(\tau-t)$. According to a usual convention, in (1.32) and (1.33) the limits of integration are extended to take into account for the possibility of impulse functions centred at the extremes. Then, a *formal definition of the fractional derivative* of positive order α could be

$$\Phi_{-\alpha} \, * \, f(t) = \frac{1}{\Gamma(-\alpha)} \int_{0^-}^{t^+} \frac{f(\tau)}{(t-\tau)^{1+\alpha}}\, d\tau\,, \quad \alpha \in \mathbb{R}^+\,.$$

The formal character is evident in that the kernel $\Phi_{-\alpha}(t)$ is not locally absolutely integrable and consequently the integral is in general divergent. In order to obtain a definition that is still valid for classical functions, we need to *regularize* the divergent integral in some way. For this purpose let us consider the integer $m \in \mathbb{N}$ such that

$m-1<\alpha<m$ and write $-\alpha=-m+(m-\alpha)$ or $-\alpha=(m-\alpha)-m$. We then obtain

$$\Phi_{-\alpha}(t)*f(t)=\Phi_{-m}(t)*\Phi_{m-\alpha}(t)*f(t)=D_t^m\circ {}_0I_t^{m-\alpha}\,f(t)\,, \quad (1.34)$$

or

$$\Phi_{-\alpha}(t)*f(t)=\Phi_{m-\alpha}(t)*\Phi_{-m}(t)*f(t)={}_0I_t^{m-\alpha}\circ D_t^m\,f(t). \quad (1.35)$$

As a consequence we derive two alternative definitions for the fractional derivative (1.34) and (1.35) corresponding to (1.13) and (1.17), respectively. The singular behaviour of $\Phi_{-m}(t)$ as a proper generalized (i.e. non-standard) function is reflected in the non-commutativity of convolution for $\Phi_{m-\alpha}(t)$ and $\Phi_m(t)$ in these formulas.

<u>Remark</u> : We recall an additional definition for the fractional derivative recently introduced by Hilfer for the order interval $0<\alpha\le 1$, see [Hilfer (2000b)], p. 113 and [Seybold and Hilfer (2005)], which interpolates the definitions (1.13) and (1.17). Like the two derivatives previously discussed, it is related to the Riemann-Liouville fractional integral. In our notation it reads

$$ {}_0D_t^{\alpha,\beta} := {}_0I_t^{\beta(1-\alpha)}\circ D_t^1\circ {}_0I_t^{(1-\beta)(1-\alpha)}\,, \quad \begin{cases} 0<\alpha\le 1\,, \\ 0<\beta\le 1\,. \end{cases} \quad (1.36)$$

We call it the *Hilfer fractional derivative* of order α and type β. The Riemann-Liouville derivative of order α corresponds to the type $\beta=0$, while the Caputo derivative to the type $\beta=1$.

1.3 Fractional relaxation equations in $\mathbb{R}^+$

The different roles played by the R-L and C fractional derivatives are more clear when the fractional generalization of the first-order differential equation governing the exponential relaxation phenomena is considered. Recalling (in non-dimensional units) the initial value problem

$$\frac{du}{dt}=-u(t)\,,\quad t\ge 0\,,\quad \text{with}\quad u(0^+)=1\,, \quad (1.37)$$

whose solution is

$$u(t) = \exp(-t)\,, \tag{1.38}$$

the following three alternatives with respect to the R-L and C fractional derivatives with $\alpha \in (0,1)$ are offered in the literature:

$${}_0^*D_t^\alpha\, u(t) = -u(t)\,, \quad t \ge 0\,, \quad \text{with} \quad u(0^+) = 1\,, \tag{1.39a}$$

$${}_0D_t^\alpha\, u(t) = -u(t)\,, \quad t \ge 0\,, \quad \text{with} \quad \lim_{t\to 0^+} {}_0I_t^{1-\alpha}\, u(t) = 1\,, \tag{1.39b}$$

$$\frac{du}{dt} = -\,{}_0D_t^{1-\alpha}\, u(t)\,, \quad t \ge 0\,, \quad \text{with} \quad u(0^+) = 1\,. \tag{1.39c}$$

In analogy with the standard problem (1.37) we solve these three problems with the Laplace transform technique, using the rules (1.27), (1.28) and (1.29), respectively. Problems (a) and (c) are *equivalent* since the Laplace transform of the solution in both cases comes out to be

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + 1}\,, \tag{1.40}$$

whereas in case (b) we get

$$\widetilde{u}(s) = \frac{1}{s^\alpha + 1} = 1 - s\,\frac{s^{\alpha-1}}{s^\alpha + 1}\,. \tag{1.41}$$

The Laplace transforms in (1.40) and (1.41) can be expressed in terms of functions of Mittag-Leffler type, of which we provide information in Appendix E. In fact, in virtue of the Laplace transform pairs (E.52) and (E.53), we have

$$\mathcal{L}\,[E_\alpha(-\lambda t^\alpha); s] = \frac{s^{\alpha-1}}{s^\alpha + \lambda},\; \mathcal{L}\{t^{\beta-1}\, E_{\alpha,\beta}(-\lambda t^\alpha); s\} = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \tag{1.42}$$

where

$$E_\alpha(-\lambda t^\alpha) := \sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n + 1)},\; E_{\alpha,\beta}(-\lambda t^\alpha) := \sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n + \beta)}, \tag{1.43}$$

with $\alpha, \beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$.

Then we obtain in the equivalent cases (a) and (c) :

$$u(t) = \psi_\alpha(t) := E_\alpha(-t^\alpha)\,, \quad t \ge 0\,, \quad 0 < \alpha < 1\,, \tag{1.44}$$

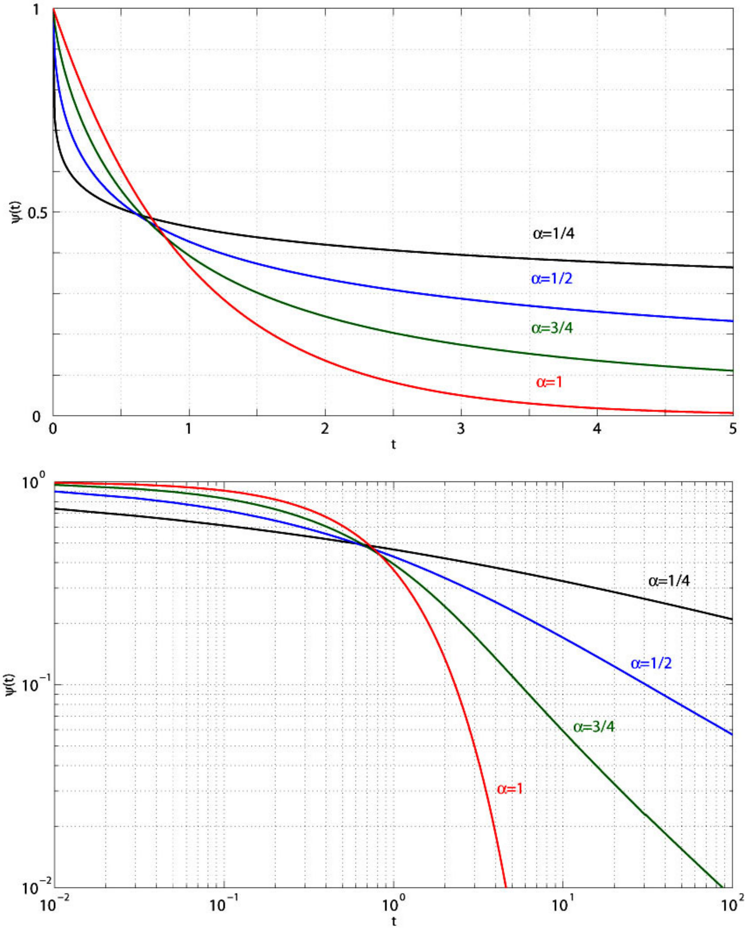

Fig. 1.1 Plots of $\psi_\alpha(t)$ with $\alpha = 1/4, 1/2, 3/4, 1$ versus t; top: linear scales ($0 \le t \le 5$); bottom: logarithmic scales ($10^{-2} \le t \le 10^2$).

and in case (b) :

$$u(t) = \phi_\alpha(t) := t^{-(1-\alpha)}\, E_{\alpha,\alpha}\left(-t^\alpha\right) \\ := -\frac{d}{dt} E_\alpha\left(-t^\alpha\right), \quad t \ge 0, \quad 0 < \alpha < 1\,. \tag{1.45}$$

The plots of the solutions $\psi_\alpha(t)$ and $\phi_\alpha(t)$ are shown in Figs. 1.1 and 1.2, respectively, for some rational values of the parameter α, by adopting linear and logarithmic scales.

It is evident that for $\alpha \to 1^-$ the solutions of the three initial value problems reduce to the standard exponential function (1.38) since in all cases $\widetilde{u}(s) \to 1/(s+1)$. However, case (b) is of minor interest from a physical view-point since the corresponding solution (1.45) is infinite in the time-origin for $0 < \alpha < 1$.

Whereas for the equivalent cases (a) and (c) the corresponding solution shows a continuous transition to the exponential function for any $t \geq 0$ when $\alpha \to 1^-$, for the case (b) such continuity is lost.

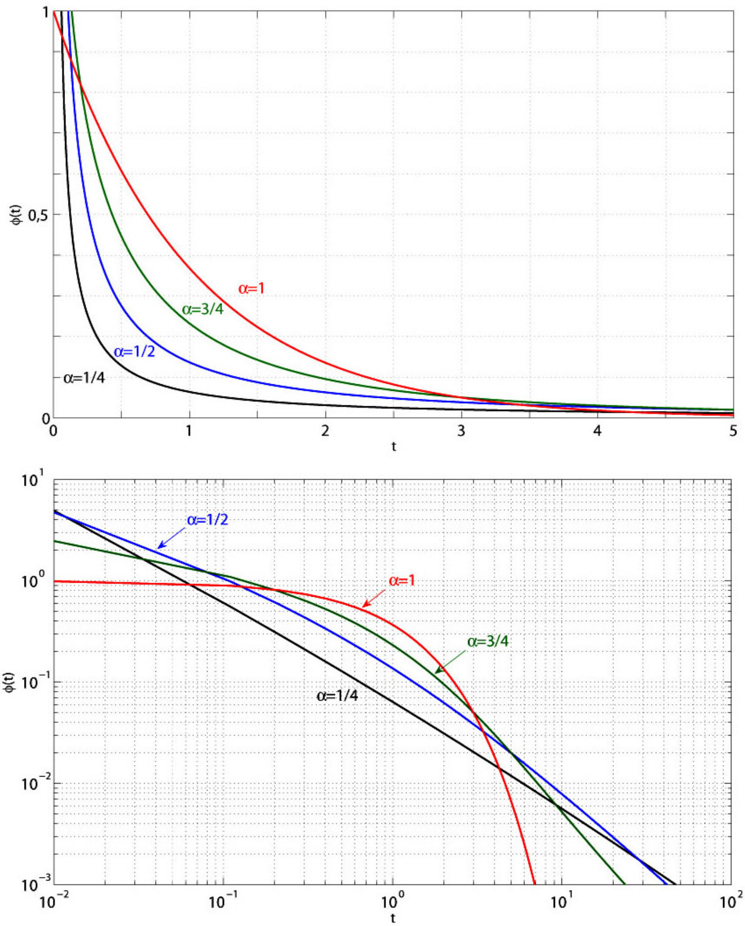

Fig. 1.2 Plots of $\phi_\alpha(t)$ with $\alpha = 1/4, 1/2, 3/4, 1$ versus t; top: linear scales $(0 \leq t \leq 5)$; bottom: logarithmic scales $(10^{-2} \leq t \leq 10^2)$.

It is worth noting the algebraic decay of $\psi_\alpha(t)$ and $\phi_\alpha(t)$ as $t \to \infty$:

$$\begin{cases} \psi_\alpha(t) \sim \dfrac{\sin(\alpha\pi)}{\pi} \dfrac{\Gamma(\alpha)}{t^\alpha}\,, \\ \\ \phi_\alpha(t) \sim \dfrac{\sin(\alpha\pi)}{\pi} \dfrac{\Gamma(\alpha+1)}{t^{(\alpha+1)}}\,, \end{cases} \quad t \to +\infty\,. \tag{1.46}$$

<u>Remark</u> : If we adopt the Hilfer intermediate derivative in fractional relaxation, that is

$${}_0D_t^{\alpha,\beta}\, u(t) = -u(t)\,,\; t \ge 0\,,\; \lim_{t\to 0^+} {}_0I_t^{(1-\alpha)(1-\beta)}\, u(t) = 1\,, \tag{1.47}$$

the Laplace transform of the solution turns out to be

$$\widetilde{u}(s) = \frac{s^{\beta(\alpha-1)}}{s^\alpha + 1}\,, \tag{1.48}$$

see [Hilfer (2000b)], so, in view of Eq. (1.43),

$$u(t) = H_{\alpha,\beta}(t) := t^{(1-\beta)(\alpha-1)}\, E_{\alpha,\alpha+\beta(1-\alpha)}(-t^\alpha)\,,\; t \ge 0\,. \tag{1.49}$$

For plots of the Hilfer function $H_{\alpha,\beta}(t)$ we refer to [Seybold and Hilfer (2005)].

1.4 Fractional integrals and derivatives with support in $\mathbb{R}$

Choosing $-\infty$ as the lower limit in the fractional integral, we have the so-called *Liouville-Weyl fractional integral.* For any $\alpha > 0$ we write

$${}_{-\infty}I_t^\alpha\, f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-\tau)^{\alpha-1}\, f(\tau)\, d\tau\,, \quad \alpha \in \mathbb{R}^+\,, \tag{1.50}$$

and consequently, we define the *Liouville–Weyl fractional derivative* of order α as its left inverse operator:

$${}_{-\infty}D_t^\alpha\, f(t) := D_t^m \circ {}_{-\infty}I_t^{m-\alpha}\, f(t)\,,\; m-1 < \alpha \le m\,, \tag{1.51}$$

with $m \in \mathbb{N}$, namely

$${}_{-\infty}D_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_{-\infty}^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, \\ \\ \dfrac{d^m}{dt^m} f(t)\,, & \alpha = m\,. \end{cases} \tag{1.51a}$$

In this case, assuming $f(t)$ to vanish as $t \to -\infty$ along with its first $m-1$ derivatives, we have the identity

$$D_t^m \circ {}_{-\infty}I_t^{m-\alpha} \, f(t) \, = \, {}_{-\infty}I_t^{m-\alpha} \, \circ \, D_t^m \, f(t) \, , \tag{1.52}$$

in contrast with (1.18). While for the Riemann–Liouville fractional integral (1.2) a sufficient condition for the convergence of the integral is given by the asymptotic behaviour

$$f\,(t) = O\left(t^{\epsilon-1}\right) \, , \; \epsilon > 0 \, , \; t \to 0^+ \, , \tag{1.53}$$

a corresponding sufficient condition for (1.50) to converge is

$$f\,(t) = O\left(|t|^{-\alpha-\epsilon}\right) \, , \quad \epsilon > 0 \, , \; t \to -\infty \, . \tag{1.54}$$

Integrable functions satisfying the properties (1.53) and (1.54) are sometimes referred to as functions of *Riemann class* and *Liouville class*, respectively, see [Miller and Ross (1993)]. For example, power functions t^γ with $\gamma > -1$ and $t > 0$ (and hence also constants) are of Riemann class, while $|t|^{-\delta}$ with $\delta > \alpha > 0$ and $t < 0$ and $\exp{(ct)}$ with $c > 0$ are of Liouville class. For the above functions we obtain

$$\begin{cases} {}_{-\infty}I_t^\alpha \, |t|^{-\delta} \, = \, \dfrac{\Gamma(\delta-\alpha)}{\Gamma(\delta)} \, |t|^{-\delta+\alpha} \, , \\[2ex] {}_{-\infty}D_t^\alpha \, |t|^{-\delta} \, = \, \dfrac{\Gamma(\delta+\alpha)}{\Gamma(\delta)} \, |t|^{-\delta-\alpha} \, , \end{cases} \tag{1.55}$$

and

$$\begin{cases} {}_{-\infty}I_t^\alpha \, \mathrm{e}^{\,ct} = c^{\,-\alpha} \, \mathrm{e}^{\,ct} \, , \\[2ex] {}_{-\infty}D_t^\alpha \, \mathrm{e}^{\,ct} = c^{\,\alpha} \, \mathrm{e}^{\,ct} \, . \end{cases} \tag{1.56}$$

Causal functions can be considered in the above integrals with the due care. In fact, in view of the possible jump discontinuities of the integrands at $t = 0$, in this case it is worthwhile to write

$$\int_{-\infty}^t (\ldots)\, d\tau = \int_{0^-}^t (\ldots)\, d\tau \, .$$

As an example we consider for $0 < \alpha < 1$ the identity

$$\frac{1}{\Gamma(1-\alpha)} \int_{0^-}^t \frac{f'(\tau)}{(t-\tau)^\alpha} \, d\tau = \frac{f(0^+)\, t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} \, d\tau \, ,$$

that is consistent with (1.19) for $m = 1$, that is

$${}_0D_t^\alpha \, f(t) = \frac{f(0^+)\, t^{-\alpha}}{\Gamma(1-\alpha)} + {}_0^*D_t^\alpha \, f(t) \, .$$

1.5 Notes

The fractional calculus may be considered an *old* and yet *novel* topic. It is an *old* topic because, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed progressively up to now. A list of mathematicians, who have provided important contributions up to the middle of the twentieth century, includes P.S. Laplace (1812), S.F. Lacroix (1819), J.B.J. Fourier (1822), N.H. Abel (1823–1826), J. Liouville (1832–1873), B. Riemann (1847), H. Holmgren (1865–1867), A.K. Grünwald (1867–1872), A.V. Letnikov (1868–1872), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892–1912), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924–1936), E.L. Post (1930), A. Zygmund (1935-1945), E.R. Love (1938-1996), A. Erdélyi (1939-1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), W. Feller (1952).

However, it may be considered a *novel* topic as well. Only since the Seventies has it been the object of specialized conferences and treatises. For the first conference the merit is due to B. Ross who, shortly after his Ph.D. dissertation on fractional calculus, organized the *First Conference on Fractional Calculus and its Applications* at the University of New Haven in June 1974, and edited the proceedings, see [Ross (1975a)]. For the first monograph the merit is ascribed to K.B. Oldham and J. Spanier, see [Oldham and Spanier (1974)] who, after a joint collaboration begun in 1968, published a book devoted to fractional calculus in 1974.

Nowadays, the series of texts devoted to fractional calculus and its applications includes over ten titles, including (alphabetically ordered by the first author) [Kilbas *et al.* (2006); Kiryakova (1994); Miller and Ross (1993); Magin (2006); Nishimoto (1991); Oldham and Spanier (1974); Podlubny (1999); Rubin (1996); Samko *et al.* (1993); West *et al.* (2003); Zaslavsky (2005)]. This list is expected to grow up in the forthcoming years. We also cite three books (still) in Russian: [Nakhushev (2003); Pskhu (2005); Uchaikin (2008)].

Furthermore, we call attention to some treatises which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus, although without explicit mention in their titles, see e.g. [Babenko (1986); Caputo (1969); Davis (1936); Dzherbashyan (1966); Dzherbashyan (1993); Erdélyi *et al.* (1953-1954); Gel'fand and Shilov (1964); Gorenflo and Vessella (1991)].

In recent years considerable interest in fractional calculus has been stimulated by the applications it finds in different areas of applied sciences like physics and engineering, possibly including fractal phenomena. In this respect A. Carpinteri and F. Mainardi have edited a collection of lecture notes entitled *Fractals and Fractional Calculus in Continuum Mechanics* [Carpinteri and Mainardi (1997)], whereas Hilfer has edited a book devoted to the applications in physics [Hilfer (2000a)]. In these books the mathematical theory of fractional calculus was reviewed by [Gorenflo and Mainardi (1997)] and by [Butzer and Westphal (2000)].

Now there are more books of proceedings and special issues of journals published that refer to the applications of fractional calculus in several scientific areas including special functions, control theory, chemical physics, stochastic processes, anomalous diffusion, rheology. Among the special issues which appeared in the last decade we mention: *Signal Processing*, Vol. 83, No. 11 (2003) and Vol. 86, No. 10 (2006); *Nonlinear Dynamics*, Vol. 29, No. 1-4 (2002) and Vol. 38, No. 1-4 (2004); *Journal of Vibration and Control*, Vol. 13, No 9-10 (2007) and Vol. 14, No 1-4 (2008); *Physica Scripta*, Vol. T136, October (2009). We also mention the electronic proceedings [Matignon and Montseny (1998)] and the recent books, edited by [Le Méhauté *et al.* (2005)], [Sabatier *et al.* (2007)], [Klages *et al.* (2008)], [Mathai and Haubold (2008)], which contain selected and improved papers presented at conferences and advanced schools, concerning various applications of fractional calculus.

Already since several years, there exist two international journals devoted almost exclusively to the subject of fractional calculus: *Journal of Fractional Calculus* (Editor-in-Chief: K. Nishimoto, Japan) started in 1992, and *Fractional Calculus and Applied Analysis* (Managing Editor: V. Kiryakova, Bulgaria) started in 1998, see `http://www.diogenes.bg/fcaa/`. Furthermore, web–sites devoted to fractional calculus have been set up, among which we call special attention to `http://www.fracalmo.org` whose name is originated by FRActional CALculus MOdelling. This web–site was set up in December 2000 by the initiative of the author and some colleagues: it contains interesting news and web–links.

Quite recently the new journal *Fractional Dynamic Systems* (`http://fds.ele-math.com/`) has been announced to start in 2010.

The reader interested in the history of fractional calculus is referred to Ross' bibliographies in [Oldham and Spanier (1974)], [Ross (1975b);(1977)] and to the historical notes contained in the textbooks and reviews already cited.

Let us recall that exhaustive tables of fractional integrals are available in the second volume of the Bateman Project devoted to Integral Transforms [Erdélyi *et al.* (1953-1954)], in Chapter $XIII$.

It is worthwhile and interesting to say here something about the commonly used naming for the types of fractional integrals and derivatives that have been discussed in this chapter. Usually names are given to honour the scientists who provided the main contributions, but not necessarily to those who first introduced the corresponding notions. Surely Liouville and then Riemann (as a student!) contributed significantly towards fractional integration and differentiation, but their notions have a history. As a matter of fact it was Abel who, in his 1823 paper [Abel (1823)], solved his celebrated integral equation by using fractional integration and differentiation of order 1/2. Three years later Abel considered the generalization to any order $\alpha \in (0,1)$ in [Abel (1826)]. So Abel, using the operators that nowadays are ascribed to Riemann and Liouville, preceded these eminent mathematicians by at least ten years. Because Riemann, like Abel, worked on the positive real semi-axis $\mathbb{R}^+$, whereas Liouville and later Weyl mainly on all of $\mathbb{R}$, we would use the names of Abel-Riemann and Liouville-Weyl for the fractional integrals with support in $\mathbb{R}^+$ and $\mathbb{R}$, respectively. However, whereas for $\mathbb{R}$ we keep the names of Liouville-Weyl, for $\mathbb{R}^+$, in order to be consistent with the existing literature, we agree to use the names of Riemann-Liouville, even if this is an injustice towards Abel.

In $\mathbb{R}$ we have not discussed the approach investigated and used in several papers by Beyer, Kempfle and Schaefer, that is appropriate for causal processes not starting at a finite instant of time, see e.g. [Beyer and Kempfle (1994); Beyer and Kempfle (1995); Kempfle (1998; Kempfle and Schäfer (1999); Kempfle and Schäfer (2000); Kempfle *et al.* (2002a); Kempfle *et al.* (2002b)]. They define the time-fractional derivative on the whole real line as a pseudo-differential operator via its Fourier symbol.

In this book, special attention is devoted to an alternative form of fractional derivative (where the orders of fractional integration and ordinary differentiation are interchanged) that nowadays is known as the *Caputo derivative.* As a matter of fact, such a form is found in a paper by Liouville himself as noted by Butzer and Westphal [Butzer and Westphal (2000)] but Liouville, not recognizing its role, disregarded this notion. As far as we know, up to to the middle of the twentieth century most authors did not take notice of the difference between the two forms and of the possible use of the alternative form. Even in the classical book on Differential and Integral Calculus by the eminent mathematician R. Courant, the two forms of the fractional derivative were considered as equivalent, see [Courant (1936)], pp. 339-341. As shown in Eqs. (1.19) and (1.20) the alternative form (denoted with the additional apex $*$) can be considered as a regularization of the *Riemann–Liouville* derivative which identically vanishes when applied to a constant. Only in the late sixties was the relevance of the alternative form recognized. In fact, in [Dzherbashyan and Nersesyan (1968)] and then in [Kochubei (1989); Kochubei (1990)] the authors used the alternative form as given by (1.19) in dealing with Cauchy problems for differential equations of fractional order. Formerly, Caputo, see [Caputo (1967); Caputo (1969)] introduced this form as given by Eq. (1.17) proving the corresponding rule in the Laplace transform domain, see Eq. (1.27). With his derivative Caputo was thus able to generalize the rule for the Laplace transform of a derivative of integer order and to solve some problems in Seismology in a proper way. Soon later, this derivative was adopted by [Caputo and Mainardi (1971a); (1971b)] in the framework of the theory of *Linear Viscoelasticity.*

Since the seventies a number of authors have re-discovered and used the alternative form, recognizing its major usefulness for solving physical problems with standard initial conditions. Although several papers by different authors appeared where the alternative derivative was adopted, it was only in the late nineties, with the tutorial paper [Gorenflo and Mainardi (1997)] and the book [Podlubny (1999)], that such form was popularized. In these references the Caputo form was named the *Caputo fractional derivative*, a term now universally accepted in the literature. The reader, however, is alerted that in a very few papers the Caputo derivative is referred to as the Caputo–Dzherbashyan derivative. Note also the transliteration as Djrbashyan.

As a relevant topic, let us now consider the question of notation. Following [Gorenflo and Mainardi (1997)] the present author opposes to the use of the notation ${}_0D_t^{-\alpha}$ for denoting the fractional integral; it is misleading, even if it is used in such distinguished treatises as [Oldham and Spanier (1974); Miller and Ross (1993); Podlubny (1999)]. It is well known that derivation and integration operators are not inverse to each other, even if their order is integer, and therefore such indiscriminate use of symbols, present only in the framework of the fractional calculus, appears unjustified. Furthermore, we have to keep in mind that for fractional order the derivative is yet an *integral* operator, so that, perhaps, it would be less disturbing to denote our ${}_0D_t^{\alpha}$ as ${}_0I_t^{-\alpha}$, than our ${}_0I_t^{\alpha}$ as ${}_0D_t^{-\alpha}$.

The notation adopted in this book is a modification of that introduced in a systematic way by [Gorenflo and Mainardi (1997)] in their CISM lectures, that, in its turn, was partly based on the book on Abel Integral Equations [Gorenflo and Vessella (1991)] and on the article [Gorenflo and Rutman (1994)].

As far as the Mittag-Leffler function is concerned, we refer the reader to Appendix E for more details, along with historical notes therein.