Chapter 3

Fractional Viscoelastic Models

Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory.

Our analysis, based on the classical linear theory of viscoelsticity recalled in Chapter 2, will start from the power law creep to justify the introduction of the operators of fractional calculus into the stress-strain relationship. So doing, we will arrive at the fractional generalization of the classical mechanical models through a correspondence principle. We will devote particular attention to the generalization of the Zener model (Standard Linear Solid) of which we will provide a physical interpretation.

We will also consider the effects of the initial conditions in properly choosing the mathematical definition for the fractional derivatives that are expected to replace the ordinary derivatives in the classical models.

3.1 The fractional calculus in the mechanical models

3.1.1 *Power-Law creep and the Scott-Blair model*

Let us consider the viscoelastic solid with *creep compliance*,

$$J(t) = \frac{a}{\Gamma(1+\nu)}\, t^{\nu}\,, \quad a > 0\,, \quad 0 < \nu < 1\,, \tag{3.1}$$

where the coefficient in front of the power-law function has been introduced for later convenience. Such creep behaviour is found to be of great interest in a number of creep experiments; usually it is referred to as the *power-law creep.* This law is compatible with the mathematical theory presented in Section 2.5, in that there exists a corresponding non-negative *retardation spectrum* (in time and frequency). In fact, by using the method of Laplace integral pairs and the reflection formula for the Gamma function,

$$\Gamma(\nu)\,\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}\,,$$

we find

$$R_\epsilon(\tau) = \frac{\sin \pi\nu}{\pi}\,\frac{1}{\tau^{1-\nu}} \iff S_\epsilon(\gamma) = a\,\frac{\sin \pi\nu}{\pi}\,\frac{1}{\gamma^{1+\nu}}\,. \tag{3.2}$$

In virtue of the *reciprocity relationship* (2.8) in the Laplace domain we can find for such viscoelastic solid its *relaxation modulus*,and then the corresponding *relaxation spectrum.* After simple manipulations we get

$$G(t) = \frac{b}{\Gamma(1-\nu)}\,t^{-\nu}\,, \quad b = \frac{1}{a} > 0\,, \tag{3.3}$$

and

$$R_\sigma(\tau) = \frac{\sin \pi\nu}{\pi}\,\frac{1}{\tau^{1+\nu}} \iff S_\sigma(\gamma) = b\,\frac{\sin \pi\nu}{\pi}\,\frac{1}{\gamma^{1-\nu}}\,. \tag{3.4}$$

For our viscoelastic solid exhibiting power-law creep, the stress-strain relationship in the *creep representation* can be easily obtained by inserting the creep law (3.1) into the integral in (2.4a). We get:

$$\epsilon(t) = \frac{a}{\Gamma(1+\nu)} \int_{-\infty}^{t} (t-\tau)^{\nu}\, d\sigma\,. \tag{3.5}$$

Writing $d\sigma = \dot\sigma(t)\,dt$ and integrating by parts, we finally have

$$\epsilon(t) = \frac{a}{\Gamma(1+\nu)} \int_{-\infty}^{t} (t-\tau)^{\nu-1}\,\sigma(\tau)\,d\tau = a\,\cdot\,{}_{-\infty}I_t^{\nu}\;[\sigma(t)]\;, \tag{3.6}$$

where ${}_{-\infty}I_t^{\nu}$ denotes the *fractional integral* of order ν with starting point $-\infty$, the so-called Liouville-Weyl integral introduced in Section 1.3.

In the *relaxation representation* the stress-strain relationship is now obtained from (2.4b) and (3.3). Writing $d\epsilon = \dot\epsilon(t)\,dt$, we get

$$\sigma(t) = \frac{b}{\Gamma(1-\nu)} \int_{-\infty}^{t} (t-\tau)^{-\nu}\,\dot\epsilon(\tau)\,d\tau = b\,\cdot\,{}_{-\infty}D_t^{\nu}\;[\epsilon(t)]\;, \tag{3.7}$$

where

$$ {}_{-\infty}D_t^{\nu} := {}_{-\infty}I_t^{1-\nu} \circ D_t = D_t \circ {}_{-\infty}I_t^{1-\nu} \,, \text{ with } D_t := \frac{d}{dt} \,, \tag{3.8} $$

denotes the *fractional derivative* of order ν with starting point $-\infty$, the so-called Liouville-Weyl derivative introduced in Section 1.4.

From now on we will consider *causal histories*, so the starting point in Eqs. (3.5)-(3.8) is 0 instead of $-\infty$. This implies that the Liouville-Weyl integral and the Liouville-Weil derivative must be replaced by the Riemann-Liouville integral ${}_0I_t^{\nu}$, introduced in Section 1.1, and by the Riemann-Liouville (R-L) or by the Caputo (C) derivative, introduced in Section 1.2, denoted respectively by ${}_0D_t^{\nu}$ and ${}_0^*D_t^{\nu}$. Later, in Section 2.5, we will show the equivalence between the two types of fractional derivatives as far as we remain in the framework of our constitutive equations and our preference for the use of fractional derivative in the Caputo sense. Thus, for causal histories, we write

$$ \epsilon(t) = a \cdot {}_0I_t^{\nu} \, [\sigma(t)] \,, \tag{3.9} $$

$$ \sigma(t) = b \cdot {}_0D_t^{\nu} \, \epsilon(t) = b \cdot {}_0^*D_t^{\nu} \, [\epsilon(t)] \,, \tag{3.10} $$

where $ab = 1$.

Some authors, e.g. [Bland (1960)], refer to Eq. (3.10) (with the R-L derivative) as the *Scott-Blair stress-strain law*. Indeed Scott-Blair was the scientist who, in the middle of the past century, proposed such a constitutive equation to characterize a viscoelastic material whose mechanical properties are intermediate between those of a pure elastic solid (Hooke model) and a pure viscous fluid (Newton model).

3.1.2 *The correspondence principle*

The use of fractional calculus in linear viscoelasticity leads us to generalize the classical mechanical models, in that the basic Newton element (dashpot) is substituted by the more general Scott-Blair element (of order ν), sometimes referred to as *pot*. In fact, we can construct the class of these generalized models from Hooke and Scott-Blair elements, disposed singly and in branches of two (in series or in parallel).

The material functions are obtained using the combination rule; their determination is made easy if we take into account the following *correspondence principle* between the classical and fractional mechanical models, as introduced in [Caputo and Mainardi (1971b)], that is empirically justified. Taking $0 < \nu \le 1$, such a correspondence principle can be formally stated by the following three equations where Laplace transform pairs are outlined:

$$\delta(t) \div 1 \Rightarrow \frac{t^{-\nu}}{\Gamma(1-\nu)} \div \frac{1}{s^{1-\nu}}, \tag{3.11}$$

$$t \div \frac{1}{s^2} \Rightarrow \frac{t^{\nu}}{\Gamma(1+\nu)} \div \frac{1}{s^{\nu+1}}, \tag{3.12}$$

$$\mathrm{e}^{-t/\tau} \div \frac{1}{s+1/\tau} \Rightarrow \mathrm{E}_\nu[-(t/\tau)^\nu] \div \frac{s^{\nu-1}}{s^\nu + (1/\tau)^\nu}, \tag{3.13}$$

where $\tau > 0$ and E_ν denotes the Mittag-Leffler function of order ν.

In Fig. 3.1, we display plots of the function $\mathrm{E}_\nu(-t^\nu)$ versus t for some (rational) values of ν.

Referring the reader to Appendix E for more details on this function, here we recall its asymptotic representations for small and large times,

$$\mathrm{E}_\nu(-t^\nu) \sim 1 - \frac{t^\nu}{\Gamma(1+\nu)}, \quad t \to 0^+; \tag{3.14}$$

$$\mathrm{E}_\nu(-t^\nu) \sim \frac{t^{-\nu}}{\Gamma(1-\nu)}, \quad t \to +\infty. \tag{3.15}$$

We easily recognize that, compared to the exponential obtained for $\nu = 1$, the *fractional* relaxation function $\mathrm{E}_\nu(-t^\nu)$ exhibits a very different behaviour. In fact, for $0 < \nu < 1$, as shown in Eqs. (3.14) and (3.15) our function exhibits for small times a much faster decay (the derivative tends to $-\infty$ in comparison with -1), and for large times a much slower decay (algebraic decay in comparison with exponential decay).

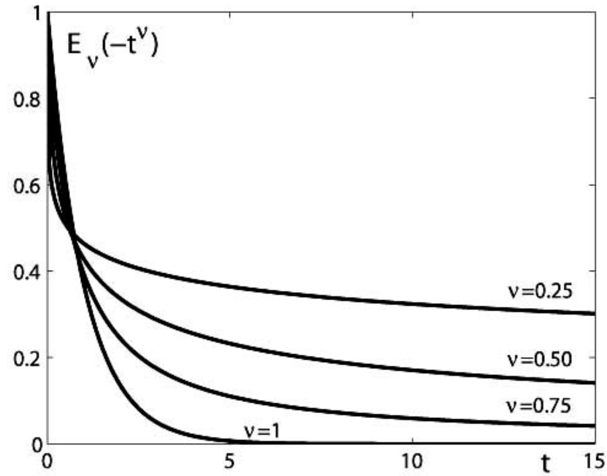

Fig. 3.1 The Mittag-Leffler function $E_\nu(-t^\nu)$ versus t ($0 \le t \le 15$) for some rational values of ν, i.e. $\nu = 0.25\,,\,0.50\,,\,0.75\,,1$.

3.1.3 *The fractional mechanical models*

We now consider the fractional generalizations of the Newton, Voigt, Maxwell, Zener and anti-Zener models. For this purpose it is sufficient to replace the derivative of order 1 with the fractional derivative of order $\nu \in (0,1)$ (in the R-L or C sense) in their constitutive equations (2.16a)-(2.20a) and then make use of the correspondence principle stated by Eqs. (3.11)-(3.13). We then obtain the following stress-strain relationships and corresponding material functions:

$$fractional\ Newton\ (Scott{-}Blair)\ model\ :\ \sigma(t) = b_1\,\frac{d^\nu \epsilon}{dt^\nu}\,, \quad (3.16a)$$

$$\begin{cases} J(t) = \dfrac{t^\nu}{b_1\,\Gamma(1+\nu)}\,, \\ G(t) = b_1\,\dfrac{t^{-\nu}}{\Gamma(1-\nu)}\,; \end{cases} \quad (3.16b)$$

$$fractional\ Voigt\ model\ :\ \sigma(t) = m\,\epsilon(t) + b_1\,\frac{d^\nu \epsilon}{dt^\nu}\,, \quad (3.17a)$$

$$\begin{cases} J(t) = \dfrac{1}{m}\left\{1 - \mathrm{E}_\nu\left[-(t/\tau_\epsilon)^\nu\right]\right\}, \\ G(t) = m + b_1\,\dfrac{t^{-\nu}}{\Gamma(1-\nu)}\,, \end{cases} \quad (3.17b)$$

where $(\tau_\epsilon)^\nu = b_1/m$;

$$fractional\ Maxwell\ model\ :\ \sigma(t) + a_1 \frac{d^\nu \sigma}{dt^\nu} = b_1 \frac{d^\nu \epsilon}{dt^\nu}\,, \qquad (3.18a)$$

$$\begin{cases} J(t) \;=\; \dfrac{a}{b_1} + \dfrac{1}{b}\, \dfrac{t^\nu}{\Gamma(1+\nu)}\,, \\ G(t) \;=\; \dfrac{b_1}{a_1}\, \mathrm{E}_\nu\left[-(t/\tau_\sigma)^\nu\right], \end{cases} \qquad (3.18b)$$

where $(\tau_\sigma)^\nu = a_1$;

$$fractional\ \ Zener\, model\ :$$

$$\left[1 + a_1 \frac{d^\nu}{dt^\nu}\right] \sigma(t) = \left[m + b_1 \frac{d^\nu}{dt^\nu}\right] \epsilon(t)\,, \qquad (3.19a)$$

$$\begin{cases} J(t) \;= J_g + J_1 \left[1 - E_\nu\left[-(t/\tau_\epsilon)^\nu\right]\right], \\ G(t) \;= G_e + G_1\, E_\nu\left[-(t/\tau_\sigma)^\nu\right], \end{cases} \qquad (3.19b)$$

where

$$\begin{cases} J_g = \dfrac{a_1}{b_1},\ J_1 = \dfrac{1}{m} - \dfrac{a_1}{b_1},\ \tau_\epsilon = \dfrac{b_1}{m}\,, \\ G_e = \ m,\ G_1 = \dfrac{b_1}{a_1} - m,\ \tau_\sigma = a_1\,; \end{cases}$$

$$fractional\ \ anti\!-\!Zener\, model\ :$$

$$\left[1 + a_1 \frac{d^\nu}{dt^\nu}\right] \sigma(t) = \left[b_1 \frac{d^\nu}{dt^\nu} + b_2 \frac{d^{(\nu+1)}}{dt^{(\nu+1)}}\right] \epsilon(t)\,, \qquad (3.20a)$$

$$\begin{cases} J(t) \;= J_+ \dfrac{t^\nu}{\Gamma(1+\nu)} + J_1\ \left[1 - \mathrm{E}_\nu\left[-(t/\tau_\epsilon)^\nu\right]\right], \\ G(t) \;= G_-\, \dfrac{t^{-\nu}}{\Gamma(1-\nu)} + G_1\, \mathrm{E}_\nu\left[-(t/\tau_\sigma)^\nu)\right], \end{cases} \qquad (3.20b)$$

where

$$\begin{cases} J_+ = \dfrac{1}{b_1},\ J_1 = \dfrac{a_1}{b_1} - \dfrac{b_2}{b_1^2},\ \tau_\epsilon = \dfrac{b_2}{b_1}, \\ G_- = \dfrac{b_2}{a_1},\ G_1 = \dfrac{b_1}{a_1} - \dfrac{b_2}{a_1^2},\ \tau_\sigma = a_1\,. \end{cases}$$

Extending the procedures of the classical mechanical models, we get the *fractional operator equation* in the form that properly generalizes Eq. (2.25):

$$\left[1+\sum_{k=1}^{p} a_k \frac{d^{\nu_k}}{dt^{\nu_k}}\right] \sigma(t) = \left[m+\sum_{k=1}^{q} b_k \frac{d^{\nu_k}}{dt^{\nu_k}}\right] \epsilon(t)\,, \tag{3.21}$$

with $\nu_k = k+\nu-1$, so, as a generalization of Eq. (2.21):

$$\begin{cases} J(t) = J_g + \sum\limits_n J_n \left\{1 - \mathrm{E}_\nu\left[-(t/\tau_{\epsilon,n})^\nu\right]\right\} + J_+ \dfrac{t^\nu}{\Gamma(1+\nu)}\,, \\ G(t) = G_e + \sum\limits_n G_n \,\mathrm{E}_\nu\left[-(t/\tau_{\sigma,n})^\nu\right] + G_- \dfrac{t^{-\nu}}{\Gamma(1-\nu)}\,, \end{cases} \tag{3.22}$$

where all the coefficients are non-negative. Of course, also for the fractional operator equation (3.21), we distinguish the same four cases of the classical operator equation (2.25), summarized in Table 2.2.

3.2 Analysis of the fractional Zener model

We now focus on the fractional Zener model. From the results for this model we can easily obtain not only those for the most simple fractional models (Scott-Blair, Voigt, Maxwell) as particular cases, but, by extrapolation, also those referring to more general models that are governed by the fractional operator equation (3.21).

3.2.1 *The material and the spectral functions*

We now consider for the fractional Zener model its creep compliance and relaxation modulus with the corresponding time-spectral functions. Following the notation of Section 2.5 we have $J(t) = J_g + J_\tau(t)$ and $G(t) = G_e + G_\tau(t)$ where

$$\begin{cases} J_\tau(t) = J_1 \left\{1 - \mathrm{E}_\nu\left[-(t/\tau_\epsilon)^\nu\right]\right\} = J_1 \displaystyle\int_0^\infty R_\epsilon(\tau)(1-\mathrm{e}^{-t/\tau})\,d\tau, \\ G_\tau(t) = G_1 \mathrm{E}_\nu\left[-(t/\tau_\sigma)^\nu\right] = G_1 \displaystyle\int_0^\infty R_\sigma(\tau)\,\mathrm{e}^{-t/\tau}\,d\tau, \end{cases} \tag{3.23}$$

with $J_1 = J_e - J_g$, $G_1 = G_g - G_e$. The creep compliance $J(t)$ and the relaxation modulus $G(t)$ are depicted in Fig 3.2 for some rational values of ν.

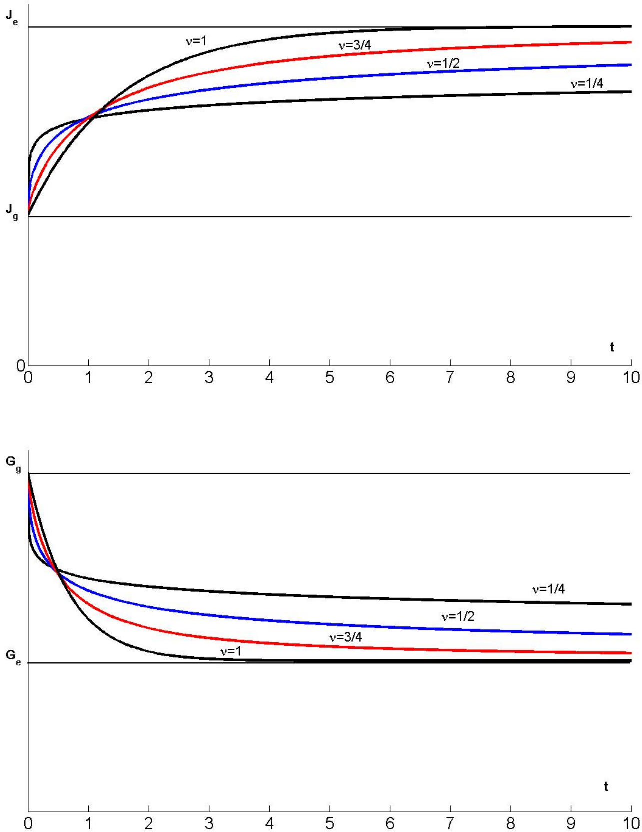

Fig. 3.2 The material functions $J(t)$ (top) and $G(t)$ (bottom) of the fractional Zener model versus t $(0 \le t \le 10)$ for some rational values of ν, i.e. $\nu = 0.25\,,\ 0.50\,,\ 0.75\,,1\,.$

Using the method of Laplace transforms illustrated in Section 2.5, we can obtain the time–spectral functions of the fractional Zener model. Denoting the suffixes $\epsilon\,,\,\sigma$ by a star, we obtain

$$R_*(\tau) = \frac{1}{\pi\,\tau}\,\frac{\sin\,\nu\pi}{(\tau/\tau_*)^{\nu} + (\tau/\tau_*)^{-\nu} + 2\,\cos\,\nu\pi}\,, \tag{3.24}$$

$$\hat{R}_*(u) = \frac{1}{2\pi} \frac{\sin \nu\pi}{\cosh \nu u + \cos \nu\pi}, \quad u = \log (\tau/\tau_*) . \tag{3.25}$$

Plots of the spectral function $R_*(\tau)$ are shown in Fig. 3.3 for some rational values of $\nu \in (0,1)$ taking $\tau_* = 1$.

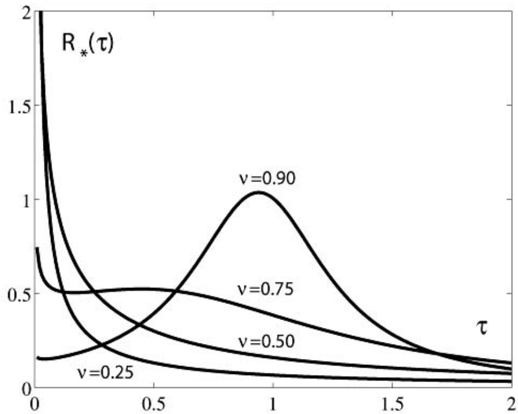

Fig. 3.3 The time–spectral function $\hat{R}_*(\tau)$ of the fractional Zener model versus τ $(0 \le \tau \le 2)$ for some rational values of ν, i.e. $\nu = 0.25\,,\, 0.50\,,\, 0.75\,, 0.90$.

From the plots of the spectra we can easily recognize the effect of a variation of ν on their character; for $\nu \to 1$ the spectra become sharper and sharper until for $\nu = 1$ they reduce to be discrete with a single retardation/relaxation time. In fact we get

$$\lim_{\nu\to 1} R_*(\tau) = \delta(\tau - 1)\,, \quad \lim_{\nu\to 1} \hat{R}_*(u) = \delta(u)\,. \tag{3.26}$$

We recognize from (3.24) that the spectrum $R_*(\tau)$ is a decreasing function of τ for $0 < \nu < \nu_0$ where $\nu_0 \approx 0.736$ is the non-zero solution of equation $\nu = \sin \nu\pi$. Subsequently, with increasing ν, it first exhibits a minimum and then a maximum before tending to the impulsive function $\delta(\tau-1)$ as $\nu \to 1$. The spectra (3.24) and (3.25) have already been calculated in [Gross (1947a)], where, in the attempt to eliminate the faults which a power law shows for the creep function, B. Gross proposed the Mittag-Leffler function as a general empirical law for both the creep and relaxation functions. Here we have newly derived this result by introducing a memory mechanism into the stress-strain relationships by means of the fractional derivative, following [Caputo and Mainardi (1971a)].

3.2.2 *Dissipation: theoretical considerations*

Let us now compute the loss tangent for the fractional Zener model starting from its complex modulus $G^*(\omega)$. For this purpose it is sufficient to properly generalize, with the fractional derivative of order ν, the corresponding formulas valid for the standard Zener model, presented in Section 2.8. Following the approach expressed by Eqs. (2.71)-(2.79), we then introduce the parameters

$$\begin{cases} \alpha := 1/\tau_\epsilon^\nu = m/b_1 \,, \\ \beta := 1/\tau_\sigma^\nu = 1/a_1 \,, \end{cases} \quad \text{with} \quad 0 < \alpha < \beta < \infty \,. \tag{3.27}$$

As a consequence, the constitutive equation (3.19a)-(3.19b) for the fractional Zener model reads

$$\left[1 + \frac{1}{\beta}\,\frac{d^\nu}{dt^\nu}\right]\sigma(t) = m\left[1 + \frac{1}{\alpha}\,\frac{d^\nu}{dt^\nu}\right]\epsilon(t)\,, \quad m = G_e = G_g\,\frac{\alpha}{\beta}\,. \tag{3.28}$$

Then, the complex modulus is

$$G^*(\omega) = G_e \frac{1 + (i\omega)^\nu/\alpha}{1 + (i\omega)^\nu/\beta} = G_g \frac{\alpha + (i\omega)^\nu}{\beta + (i\omega)^\nu}\,, \tag{3.29}$$

henceforth,

$$G^*(\omega) = G'(\omega) + G''(\omega)\,, \text{ with } \begin{cases} G'(\omega) = G_g \dfrac{\omega^2 + \alpha\,\beta}{\omega^2 + \beta^2}\,, \\ G''(\omega) = G_g \dfrac{\omega(\beta - \alpha)}{\omega^2 + \beta^2}\,. \end{cases} \tag{3.30}$$

Finally, the loss tangent is obtained from the known relationship (2.49)

$$\tan\,\delta(\omega) = \frac{G''(\omega)}{G'(\omega)}\,.$$

Then we get:

$$\begin{aligned} &\textit{fractional Zener model} \,: \\ &\tan\,\delta(\omega) = (\beta - \alpha)\,\frac{\omega^\nu \sin(\nu\pi/2)}{\omega^{2\nu} + \alpha\,\beta + (\alpha + \beta)\,\omega^\nu\,\cos(\nu\pi/2)}\,. \end{aligned} \tag{3.31}$$

For consistency of notations such expression would be compared with (2.75) rather than with (2.70), both valid for the Zener model.

As expected, from Eq. (3.31) we easily recover the expressions of the loss tangent for the limiting cases of the fractional Zener model, that is the loss tangent for the Scott-Blair model (intermediate between the Hooke and Newton models), and for the fractional Voigt and Maxwell models. We obtain:

$$\begin{array}{c} fractional\ Newton\ \ Scott{-}Blair\ model \quad (0 = \alpha < \beta = \infty)\ : \\ \tan\delta(\omega) = \tan\left(\nu\pi/2\right); \end{array} \tag{3.32}$$

$$\begin{array}{c} fractional\ \ Voigt\ model \quad (0 < \alpha < \beta = \infty)\ : \\ \tan\delta(\omega) = \dfrac{\omega^\nu\ \sin(\nu\pi/2)}{\alpha + \omega^\nu\ \cos(\nu\pi/2)}\,, \end{array} \tag{3.33}$$

$$\begin{array}{c} fractional\ \ Maxwell\ model \quad (0 = \alpha < \beta < \infty)\ : \\ \tan\delta(\omega) = \dfrac{\beta\,\omega^\nu\ \sin(\nu\pi/2)}{\omega^{2\nu} + \beta\,\omega^\nu\ \cos(\nu\pi/2)}\,. \end{array} \tag{3.34}$$

We note that the Scott-Blair model exhibits a constant loss tangent, that is, quite independent of frequency, a noteworthy property that can be used in experimental checks when ν is sufficiently close to zero. As far as the fractional Voigt and Maxwell models $(0 < \nu < 1)$ are concerned, note that the dependence of loss tangent of frequency is similar but more moderate than those for the standard Voigt and Maxwell models $(\nu = 1)$ described in Eqs. (2.78), (2.79) respectively. The same holds for the fractional Zener model in comparison with the corresponding standard model described in Eq. (2.75).

Consider again the fractional Zener model. Indeed, in view of experimental checks for viscoelastic solids exhibiting a low value for the loss tangent, say less than 10^{-2}, we find it reasonable to approximate the exact expression (3.31) of the loss tangent for the fractional Zener model as follows:

$$\tan\,\delta(\omega) \simeq (\beta - \alpha)\,\frac{\omega^\nu \sin\left(\nu\pi/2\right)}{\omega^{2\nu} + \alpha^2 + 2\alpha\,\omega^\nu\ \sin\left(\nu\pi/2\right)}\,. \tag{3.35}$$

This approximation is well justified as soon as the condition

$$\Delta := \frac{\beta - \alpha}{\alpha} \ll 1 \tag{3.36}$$

is satisfied, corresponding to the so-called *nearly elastic* case of our model, in analogy with the standard Zener model (S.L.S.). In such approximation we set

$$\begin{cases} \omega_0^\nu = \alpha \\ \Delta = \dfrac{\beta - \alpha}{\alpha} \simeq \dfrac{\beta - \alpha}{\sqrt{\alpha\,\beta}}\,, \end{cases} \tag{3.37}$$

so that

$$\tan\delta(\omega) \simeq \Delta\, \frac{(\omega/\omega_0)^\nu\, \sin(\nu\pi/2)}{1 + (\omega/\omega_0)^{2\nu} + 2\,(\omega/\omega_0)^\nu\, \cos(\nu\pi/2)}\,. \tag{3.38}$$

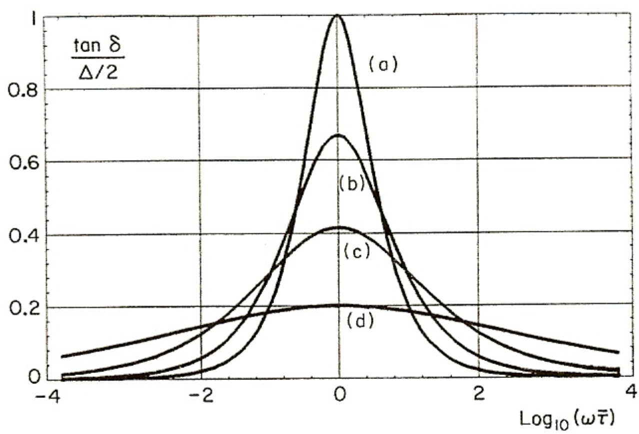

Fig. 3.4 Plots of the loss tangent $\tan\delta(\omega)$ scaled with $\Delta/2$ against the logarithm of $\omega\overline{\tau}$, for some rational values of ν: *a)* $\nu = 1$, *b)* $\nu = 0.75$, *c)* $\nu = 0.50$, *d)* $\nu = 0.25$.

It is easy to recognize that ω_0 is the frequency at which the loss tangent (3.34) assumes its maximum given by

$$\tan\delta(\omega)\big|_{max} = \frac{\Delta}{2}\, \frac{\sin(\nu\pi/2)}{1 + \cos(\nu\pi/2)}\,. \tag{3.39}$$

It may be convenient to replace in (3.38) the peak frequency ω_0 with $1/\overline{\tau}$ where $\overline{\tau}$ is a characteristic time intermediate between τ_ϵ and τ_σ. In fact, in the approximation $\alpha \simeq \beta$ we get from (3.27)

$$\omega_0 := 1/\tau_\epsilon \simeq 1/\tau_\sigma \simeq 1/\sqrt{\tau_\epsilon\, \tau_\sigma}\,. \tag{3.40}$$

Then, in terms of $\overline{\tau}$, the loss tangent in the nearly elastic approximation reads

$$\tan\delta(\omega) \simeq \Delta\, \frac{(\omega\,\overline{\tau})^\nu\, \sin(\nu\pi/2)}{1 + (\omega\,\overline{\tau})^{2\nu} + 2\,(\omega\,\overline{\tau})^\nu\, \cos(\nu\pi/2)}\,. \tag{3.38'}$$

When the loss tangent is plotted against the logarithm of $\omega/\omega_0 = \omega\overline{\tau}$, it is seen to be a symmetrical function around its maximum value attained at $\omega/\omega_0 = \omega\overline{\tau} = 1$, as shown in Fig 3.4 for some rational values of ν and for fixed Δ. We note that the peak decreases in amplitude and broadens with a rate depending on ν; for $\nu = 1$ we recover the classical Debye peak of the classical Zener solid.

For the sake of convenience, in view of applications to experimental data, in Fig. 3.5 we report the normalized loss tangent obtained when the maximum amplitude is kept constant, for the previous rational values of ν.

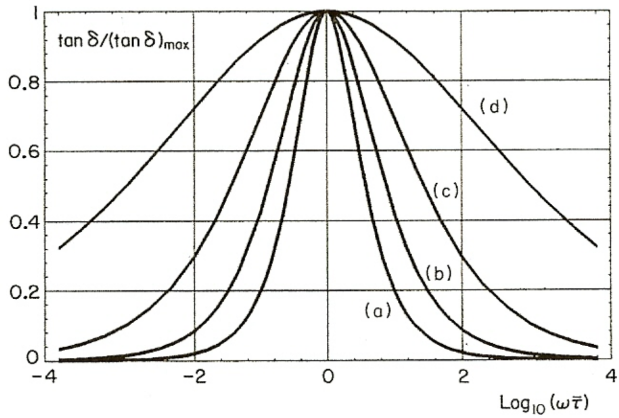

Fig. 3.5 Plots of the loss tangent $\tan\delta(\omega)$ scaled with it maximum against the logarithm of $\omega\overline{\tau}$, for some rational values of ν: $a)\,\nu = 1$, $b)\,\nu = 0.75$, $c)\,\nu = 0.50$, $d)\,\nu = 0.25$.

3.2.3 *Dissipation: experimental checks*

Experimental data on the loss tangent are available for various viscoelastic solids; however, measurements are always affected by considerable errors and, over a large frequency range, are scarce because of considerable experimental difficulties. In experiments one prefers to adopt the term *specific dissipation function* Q^{-1} rather than loss tangent, assuming they are equivalent as discussed in Section 2.7, see Eqs. (2.62)-(2.63). We also note that indirect methods of measuring the specific dissipation are used as those based on free oscillations and resonance phenomena, see e.g. [Kolsky (1953); Zener (1948)]. By these methods [Bennewitz-Rotger (1936), (1938)] measured the Q for transverse vibrations in reeds of several metals in the frequency range of three decades. Their data were fitted in [Caputo and Mainardi (1971b)] by using the expression (3.38) in view of the low values of dissipation. Precisely, in their attempt, Caputo and Mainardi computed a fit of (3.38) to the experimental curves by using the parameters Δ, α, ν as follows. From each datum they found ω_0, Q_{max}^{-1} then, (3.39) is a relationship between Δ and ν. The theoretical curve, forced to pass through the maximum of the experimental curve, was then fitted to this by using the other free parameter.

Herewith we report only the fits obtained for brass and steel, as shown in Figs. 3.6 and 3.7, respectively, where a dashed line is used for the experimental curves and a continuous line for the theoretical ones. The values of the parameter ν are listed in Table 3.1.

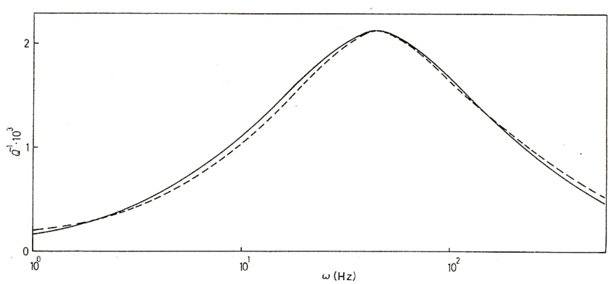

Fig. 3.6 Q^{-1} in brass: comparison between theoretical (continuous line) and experimental (dashed line) curves.

Metal	$\Delta\,(s^{-\nu})$	$\alpha\,(s^{-\nu})$	ν	$f_{max}\,(Hz)$	Q_{max}^{-1}
brass	0.77	153.2	0.90	42.7	$2.14 \cdot 10^{-3}$
steel	0.19	54.3	0.80	23.4	$1.35 \cdot 10^{-3}$

Table 3.1 Parameters for the data fit after Bennewitz and Rotger.

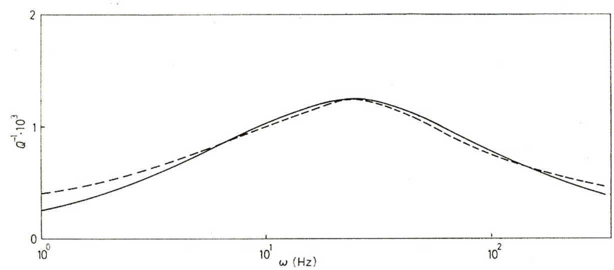

Fig. 3.7 Q^{-1} in steel: comparison between theoretical (continuous line) and experimental (dashed line) curves.

3.3 The physical interpretation of the fractional Zener model via fractional diffusion

According to [Zener (1948)] the physical interpretation of anelasticity in metals is linked to a spectrum of relaxation phenomena. In particular, the thermal relaxation due to diffusion in the thermoelastic coupling is essential to derive the standard constitutive equation (stress-strain relationship) in linear viscoelasticity. This equation corresponds to a simple rheological model (with three independent parameters) known also as Standard Linear Solid (S.L.S.), discussed in Section 2.4, see Eqs. (2.19a)-(2.19b), and in Section 2.8. We now re-write its constitutive equation in the form

$$\sigma + \tau_\epsilon \frac{d\sigma}{dt} = M_r \left(\epsilon + \tau_\sigma \frac{d\epsilon}{dt}\right) , \tag{3.41}$$

where $\sigma = \sigma(t)$ and $\epsilon = \epsilon(t)$ denote the uni-axial stress and strain respectively. The three parameters are M_r, which represents the relaxed modulus, and τ_σ , τ_ϵ , which denote the relaxation times under constant stress and strain respectively; an additional parameter is the unrelaxed modulus M_u given by $\tau_\sigma/\tau_\epsilon = M_u/M_r > 1$.

Following Zener, the model equation (3.41) can be derived from the basic equations of the thermoelastic coupling, provided that τ_σ and τ_ϵ also represent the relaxation times for temperature relaxation at constant stress and strain, respectively, and M_r and M_u represent the isothermal and adiabatic moduli, respectively.

Denoting by ΔT the deviation of the temperature from its standard value, the two basic equations of thermoelasticity are

$$\epsilon = \frac{1}{M_r}\,\sigma + \lambda \Delta T\,, \tag{3.42}$$

$$\frac{d}{dt}\Delta T = -\frac{1}{\tau_\epsilon}\,\Delta T - \gamma\,\frac{d\epsilon}{dt}\,, \tag{3.43}$$

where λ is the linear thermal expansion coefficient and $\gamma = (\partial T/\partial \epsilon)_{adiab}$. Equation (3.43) results from the combination of the two basic phenomena which induce temperature changes, (a) relaxation due to diffusion

$$\left(\frac{d}{dt}\Delta T\right)_{diff} = -\frac{1}{\tau_\epsilon}\,\Delta T\,, \tag{3.44}$$

and (b) adiabatic strain change

$$\left(\frac{d}{dt}\Delta T\right)_{adiab} = -\gamma\,\frac{d\epsilon}{dt}\,. \tag{3.45}$$

Putting $1 + \lambda\,\gamma = \tau_\sigma/\tau_\epsilon = M_u/M_r$ and eliminating ΔT between (3.42) and (3.43), the relationship (3.41) is readily obtained. In this way the temperature plays the role of a hidden variable.

If now we assume, following [Mainardi (1994b)], that the relaxation due to diffusion is of long memory type and just governed by the fractional differential equation

$$\left(\frac{d^\nu}{dt^\nu}\Delta T\right)_{diff} = -\frac{1}{\overline{\tau}_\epsilon^{\,\nu}}\,\Delta T\,, \quad 0 < \nu \le 1\,, \tag{3.46}$$

where $\overline{\tau}_\epsilon$ is a suitable relaxation time, we allow for a natural generalization of the simple process of relaxation, which now depends on the parameter ν, see e.g. [Mainardi (1996b); Mainardi (1997)]. As a consequence, Eq. (3.43) turns out to be modified into

$$\frac{d^\nu}{dt^\nu}\,\Delta T = -\frac{1}{\overline{\tau}_\epsilon^{\,\nu}}\,\Delta T - \gamma\frac{d^\nu \epsilon}{dt^\nu}\,, \tag{3.47}$$

and, *mutatis mutandis*, the stress-strain relationship turns out to be

$$\sigma + \overline{\tau}_\epsilon^{\,\nu}\,\frac{d^\nu \sigma}{dt^\nu} = M_r\,\left(\epsilon + \overline{\tau}_\sigma^{\,\nu}\,\frac{d^\nu \epsilon}{dt^\nu}\right)\,, \tag{3.48}$$

where we have used $1 + \lambda\,\gamma = (\overline{\tau}_\sigma/\overline{\tau}_\epsilon)^\nu = M_u/M_r$. So doing, we have obtained the so-called fractional Zener model, analysed in Section 3.2.

3.4 Which type of fractional derivative? Caputo or Riemann-Liouville?

In the previous sections we have investigated some physical and mathematical aspects of the use of fractional calculus in linear viscoelasticity. We have assumed that our systems are at rest for time $t < 0$. As a consequence, there is no need for including the treatment of *pre-history* as it is required in the so-called *initialised fractional calculus*, recently introduced by [Lorenzo and Hartley (2000)] and [Fukunaga (2002)].

We note that the initial conditions at $t = 0^+$ for the stress and strain do not explicitly enter into the fractional operator equation (3.21) if they are taken in the same way as for the classical mechanical models reviewed in the previous chapter (see the remark at the end of Section 2.4). This means that the approach with the Caputo derivative, which requires in the Laplace domain the same initial conditions as the classical models, is quite correct.

On the other hand, assuming the same initial conditions, the approach with the Riemann-Liouville derivative is expected to provide the same results. In fact, in view of the corresponding Laplace transform rule (1.29) for the R-L derivative, the initial conditions do not appear in the Laplace domain. Under such conditions the two approaches appear equivalent.

The equivalence of the two approaches has been noted for the fractional Zener model in a recent note by [Bagley (2007)]. However, for us the adoption of the Caputo derivative appears to be the most suitable choice, since it is fully compatible with the classical approach. We shall return to this matter in Chapter 6, when we consider wave propagation in the Scott-Blair model.

The reader is referred to [Heymans and Podlubny (2006)] for the physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville derivatives, especially in viscoelasticity.

3.5 Notes

During the twentieth-century a number of authors have (implicitly or explicitly) used the fractional calculus as an empirical method of describing the properties of viscoelastic materials. In the first half of that century the early contributors were: Gemant in USA, see [Gemant (1936); (1938)], Scott-Blair in England, see [Scott-Blair (1944); (1947); (1949)], Gerasimov and Rabotnov in the former Soviet Union, see [Gerasimov (1948)], [Rabotnov (1948)].

Gemant published a series of 16 articles entitled *Frictional Phenomena* in Journal of Applied Physics since 1941 to 1943, which were collected in a book of the same title [Gemant (1950)]. In his eighth chapter-paper [Gemant (1942)], p. 220, he referred to his previous articles [Gemant (1936); (1938))] for justifying the necessity of fractional differential operators to compute the shape of relaxation curves for some elasto-viscous fluids. Thus, the words fractional and frictional were coupled, presumably for the first time, by Gemant.

Scott-Blair used the fractional calculus approach to model the observations made by [Nutting (1921); (1943); (1946)] that the stress relaxation phenomenon could be described by fractional powers of time. He noted that time derivatives of fractional order would simultaneously model the observations of Nutting on stress relaxation and those of Gemant on frequency dependence. It is quite instructive to cite some words by Scott-Blair quoted in [Stiassnie (1979)]:

I was working on the assessing of firmness of various materials (e.g. cheese and clay by experts handling them) these systems are of course both elastic and viscous but I felt sure that judgments were made not on an addition of elastic and viscous parts but on something in between the two so I introduced fractional differentials of strain with respect to time. Later, in the same letter Scott-Blair added: *I gave up the work eventually, mainly because I could not find a definition of a fractional differential that would satisfy the mathematicians.*

The 1948 the papers by Gerasimov and Rabotnov were published in Russian, so their contents remained unknown to the majority of western scientists up to the translation into English of the treatises by Rabotnov, see [Rabotnov (1969); (1980)]. Whereas Gerasimov explicitly used a fractional derivative to define his model of viscoelasticity (akin to the Scott-Blair model), Rabotnov preferred to use the Volterra integral operators with weakly singular kernels that could be interpreted in terms of fractional integrals and derivatives. After the appearance of the books by Rabotnov it has became common to speak about *Rabotnov's theory of hereditary solid mechanics.* The relation between Rabotnov's theory and the models of fractional viscoelasticity has been briefly recalled in the recent paper [Rossikhin and Shitikova (2007)]. According to these Russian authors, Rabotnov could express his models in terms of the operators of the fractional calculus, but he considered these operators only as some mathematical abstraction.

In the late sixties, formerly Caputo, see [Caputo (1966); (1967); (1969)], then Caputo and Mainardi, see [Caputo and Mainardi (1971a); (1971b)], explicitly suggested that derivatives of fractional order (of Caputo type) could be successfully used to model the dissipation in seismology and in metallurgy. In this respect the present author likes to recall a correspondence carried out between himself (as a young post-doc student) and the Russian Academician Rabotnov, related to two courses on Rheology held at CISM (International Centre for Mechanical Sciences, Udine, Italy) in 1973 and 1974, where Rabotnov was an invited speaker but without participating, see [Rabotnov (1973); (1974)]. Rabotnov recognized the relevance of the review paper [Caputo and Mainardi (1971b)], writing in his unpublished 1974 CISM Lecture Notes:

That's way it was of great interest for me to know the paper of Caputo and Mainardi from the University of Bologna published in 1971. These authors have obtained similar results independently without knowing the corresponding Russian publications..... Then he added: *The paper of Caputo and Mainardi contains a lot of experimental data of different authors in support of their theory. On the other hand a great number of experimental curves obtained by Postnikov and his coworkers as also by foreign authors can be found in numerous papers of Shermergor and Meshkov.*

Unfortunately, the eminent Russian scientist did not cite the 1971 paper by Caputo and Mainardi (presumably for reasons independently from his willing) in the Russian and English editions of his later book [Rabotnov (1980)].

Nowadays, several articles (originally in Russian) by Shermergor, Meshkov and their associated researchers have been re-printed in English in Journal of Applied Mechanics and Technical Physics (English translation of Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki), see e.g. [Shermergor (1966)], [Meshkov *et al.* (1966)], [Meshkov (1967)], [Meshkov and Rossikhin (1968)],[Meshkov (1970)], [Zelenev *et al.* (1970)], [Gonsovskii and Rossikhin (1973)], available at the URL: `http://www.springerlink.com/`. On this respect we cite the recent review papers [Rossikhin (2010)], [Rossikhin and Shitikova (2010)] where the works of the Russian scientists on fractional viscoelasticity are examined.

The beginning of the modern applications of fractional calculus in linear viscoelasticity is generally attributed to the 1979 PhD thesis by Bagley (under supervision of Prof. Torvik), see [Bagley (1979)], followed by a number of relevant papers, e.g. [Bagley and Torvik (1979); (1983a); (1983b)] and [Torvik and Bagley (1984)]. However, for the sake of completeness, one would recall also the 1970 PhD thesis of Rossikhin under the supervision of Prof. Meshkov, see [Rossikhin (1970)], and the 1971 PhD thesis of the author under the supervision of Prof. Caputo, summarized in [Caputo and Mainardi (1971b)].

To date, applications of fractional calculus in linear and nonlinear viscoelasticity have been considered by a great and increasing number of authors to whom we have tried to refer in our huge (but not exhaustive) bibliography at the end of the book.