LECTURE NOTES ON MATHEMATICAL PHYSICS
Department of Physics, University of Bologna

ABEL INTEGRAL EQUATIONS

An Introduction via Laplace Transform and Fractional Calculus

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Fifth Edition - February 2012

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1. Basic Definitions

The subject of integral equations was started from a study by Niels Henrick Abel in the early 19th century in connection with the mechanics problem of calculating the time a particle takes to slide under gravity down a given smooth curve, from any point on the curve to its lower end. In his study Abel illustrated the elegance and power of the *fractional calculus.*

After Abel the study of integral equations was continued thanks to eminent mathematicians including V. Volterra, E.I. Fredholm, D. Hilbert, E. Schmidt, H.T. Davis and F.G. Tricomi. Here we limit ourselves to consider a particular class referred to as Abel-type equations, after having briefly recalled the basic classification of integral equations, in which integration is with respect to a single real variable.

Usually the classification of integral equations centres on three basic characteristics which together describe their overall structure.

(i)   The historical descriptions *Fredholm* and *Volterra* are concerned with the integration interval. In a Fredholm equation the integral is over a finite interval with fixed end-points; in Volterra equation the upper end-point is indefinite.

(ii)   The *kind* of an equation refers to the location of the unknown function. *First* kind equations have the unknown function present under the integral sign only; *second* kind equations also have the unknown function outside the integral.

(iii)   The adjective *singular* is sometimes used when the integration is improper, either because the interval is infinite, or because the integrand is unbounded within the given interval. Obviously an integral equation can be singular on both counts.

The unknown function will be denoted by $u = u(t)$. Every integral equation contains, in addition to a known function $f(t)$ called the *free term*, a function obtained from u by integration and of the form

$$\int_a^b K(t,\tau)\,u(\tau)\,d\tau\,, \quad \text{or} \quad \int_a^t K(t,\tau)\,u(\tau)\,d\tau\,, \quad (a \le t \le b)\,,$$

where K is called the *kernel* and is assumed known.

The Volterra/Fredholm equations of the first kind are

$$\boxed{\int_a^{t,b} K(t,\tau)\,u(\tau)\,d\tau = f(t)\,, \quad (a \le t \le b)}\,, \tag{1.1}$$

and the Volterra/Fredholm equations of second kind are

$$\boxed{u(t) + \lambda \int_a^{t,b} K(t,\tau)\,u(\tau)\,d\tau = f(t)\,, \quad \lambda \in \mathbb{C}\,, \quad (a \le t \le b)}\,. \tag{1.2}$$

The quantity λ appearing in (1.2) is a numerical parameter, generally complex. It plays a crucial part in the theory of (1.2); in practical applications λ is usually composed of physical quantities and cannot simply absorbed into the kernel.

We should also note that Fredholm equations reduce to those of Volterra type if their kernels are defined to have the property that

$$K(t,\tau) = 0 \quad (a \le t \le \tau \le b) \,. \tag{1.3}$$

This relationship between the two varieties of equation is a useful one, but it is wrong to infer that the differences between them are minimal.

A very important class of integral equations in Physics is represented by Volterra equations of *convolution* type for which $a = 0$ and the kernel satisfies the condition

$$\boxed{K(t,\tau) = K(t-\tau)} \,. \tag{1.4}$$

Consequently, the integral in (1.1-2) can be interpreted as a convolution between two *causal* functions $K(t)$ and $u(t)$ in that

$$\boxed{K(t) * u(t) := \int_{-\infty}^{+\infty} K(t-\tau)\, u(\tau)\, d\tau = \int_0^t K(t-\tau)\, u(\tau)\, d\tau} \,, \tag{1.5}$$

being $u(t) = K(t) = 0$ for $t < 0$. In this case a suited method of solution is given by the Laplace transform in that the transform of the convolution between two functions turns out to be the product of the transformed functions. Adopting the following notation

$$\mathcal{L}[f(t)] := \int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt = \widetilde{f}(s) \div f(t)\,, \quad s \in \mathbb{C}\,,$$

we write

$$\boxed{\mathcal{L}[K(t) * u(t)] = \widetilde{K}(s)\, \widetilde{u}(s) \div K(t) * u(t)} \,. \tag{1.6}$$

We define *Abel* type integral equations the Volterra equations of convolution type for which the kernel turns out to be

$$\boxed{K(t) = \frac{1}{\Gamma(\alpha)}\, \frac{1}{t^{1-\alpha}}\,, \quad t > 0\,, \quad 0 < \alpha < 1} \,. \tag{1.7}$$

The constant factor related to the Gamma function is put for convenience in order to identify the kernel with the *causal* power function

$$\boxed{\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}} \,, \tag{1.8}$$

that for $\alpha > 0$ we refer to as the (ordinary) *Gel'fand-Shilov function*.

In (1.8) the suffix $+$ is just denoting that the function is vanishing for $t < 0$. Being $\alpha > 0$, this function turns out to be *locally* absolutely integrable in $\mathbb{R}^+$. Let us recall the noteworthy property of the (ordinary) Gel'fand-Shilov functions with respect to the convolution, *i.e.*

$$\boxed{\Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t) \,, \quad \alpha \,, \beta > 0} \,. \tag{1.9}$$

More generally the Abel type integral equations are defined by a so-called *weakly singular* kernel

$$K(t,\tau) = \frac{\hat{K}(t,\tau)}{|t-\tau|^\gamma} \,, \quad 0 < \gamma < 1 \,, \tag{1.10}$$

where $\hat{K}$ is a bounded function. Special consideration needs for the case $\gamma = 1$ in (1.10), for which the kernel is termed *strongly singular* or *Cauchy singular* kernel and the corresponding integral equations are called of Cauchy type.

2. Abel Integral Equations via Laplace Transform

Let us consider the Abel integral equation of the *first kind*

$$\boxed{\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = f(t) \,, \quad 0 < \alpha < 1 \,,} \tag{2.1}$$

and of the *second kind*

$$\boxed{u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = f(t) \,, \quad \lambda \in \mathbb{C} \,, \quad 0 < \alpha < 1 \,.} \tag{2.2}$$

Let us solve the above equations making use of the Laplace transform recalling that

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = \Phi_\alpha(t) * u(t) \,, \quad \Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \div \frac{1}{s^\alpha} \,, \quad 0 < \alpha < 1 \,. \tag{2.3}$$

For the Abel equation of the *first kind* we obtain

$$\frac{\widetilde{u}(s)}{s^\alpha} = \widetilde{f}(s) \implies \widetilde{u}(s) = s^\alpha \, \widetilde{f}(s) \,. \tag{2.4}$$

Now we can choose two different ways to get the inverse Laplace transform from (2.4), according to the standard rules. We recall that the Laplace transform of any (ordinary) function must necessarily tend to 0 as $\mathrm{Re}\,\{s\} \to +\infty$. Writing (2.4) as

$$\widetilde{u}(s) = s \left[\frac{\widetilde{f}(s)}{s^{1-\alpha}} \right] \,, \tag{2.4a}$$

we obtain

$$\boxed{u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} \, d\tau} \,. \tag{2.5a}$$

If we write (2.4) as

$$\widetilde{u}(s) = \frac{1}{s^{1-\alpha}} \left[s\, \widetilde{f}(s) - f(0^+) \right] + \frac{f(0^+)}{s^{1-\alpha}} \,, \qquad (2.4b)$$

we obtain

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau + f(0^+) \frac{t^{-\alpha}}{\Gamma(1-a)} \,. \qquad (2.5b)$$

The way *b)* requires that $f(t)$ be differentiable with $\mathcal{L}$-transformable derivative; consequently $0 \le |f(0^+)| < \infty$. Then it turns out from (2.5b) that $u(0^+)$ can be infinite if $f(0^+) \ne 0$, being $u(t) = O(t^{-\alpha})$. The way *a)* requires weaker conditions in that the integral at the R.H.S. of (2.5a) must be vanishing as $t \to 0^+$; consequently $f(0^+)$ could be infinite but with $f(t) = O(t^{-\nu})$, $0 < \nu < 1 - \alpha$ (keeping in mind that $\Phi_{1-\alpha} * \Phi_{1-\nu} = \Phi_{2-\alpha-\nu}$). Then it turns out from (2.5a) that $u(0^+)$ can be infinite if $f(0^+)$ is infinite, being $u(t) = O(t^{-(\alpha+\nu)})$.

For the Abel equation of the *second kind* we obtain

$$\left[1 + \frac{\lambda}{s^\alpha} \right] \widetilde{u}(s) = \widetilde{f}(s) \implies \widetilde{u}(s) = \frac{s^\alpha}{s^\alpha + \lambda}\, \widetilde{f}(s)\,. \qquad (2.6)$$

Now let us introduce for $t \ge 0$ the function $e_\alpha(t; \lambda)$ so defined

$$e_\alpha(t; \lambda) := E_\alpha(-\lambda t^\alpha) \; \div \; \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \qquad \lambda \in \mathbb{C}\,, \qquad (2.7)$$

where E_α denotes the Mittag-Leffler function of order α. We recall that the Mittag-Leffler function is defined for any positive α in the whole complex plane as

$$E_\alpha(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}\,, \qquad \alpha > 0\,, \quad z \in \mathbb{C}\,. \qquad (2.8)$$

As before, we can choose two different ways to get the inverse Laplace transform from (2.6), according to the standard rules. Writing (2.6) as

$$\widetilde{u}(s) = s \left[\frac{s^{\alpha-1}}{s^\alpha + \lambda}\, \widetilde{f}(s) \right] \,, \qquad (2.6a)$$

we obtain

$$u(t) = \frac{d}{dt} \int_0^t e_\alpha(t - \tau; \lambda)\, f(\tau)\, d\tau \,. \qquad (2.9a)$$

If we write (2.6) as

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda} \left[s\, \widetilde{f}(s) - f(0^+) \right] + f(0^+) \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \qquad (2.6b)$$

we obtain

$$u(t) = \int_0^t e_\alpha(t - \tau; \lambda)\, f'(\tau)\, d\tau + f(0^+)\, e_\alpha(t; \lambda) \,. \qquad (2.9b)$$

We also note that, being $e_\alpha(t;\lambda)$ a function differentiable with respect to t with $e_\alpha(0^+;\lambda) = E_\alpha(0^+) = 1$, it exists another possibility to re-write (2.6), namely

$$\widetilde{u}(s) = \left[s\, \frac{s^{\alpha-1}}{s^\alpha + \lambda} - 1 \right] \widetilde{f}(s) + \widetilde{f}(s)\,. \tag{2.9c}$$

Then we obtain

$$\boxed{u(t) = \int_0^t e_\alpha'(t - \tau;\lambda)\, f(\tau)\, d\tau + f(t)}\,. \tag{2.8c}$$

We recognize that the way *b)* is more restrictive than the ways *a)* and *c)* since it requires that $f(t)$ be differentiable with $\mathcal{L}$-transformable derivative.

We conclude this section with some properties of the Mittag-Leffler type function $e_\alpha(t;\lambda)$ for $\underline{0 < \alpha < 1}$. First of all one notes the explicit expression in terms of the *complementary error function* that our function assumes in the particular case $\alpha = 1/2$, for any $\lambda \in \mathbb{C}$. In fact one can easily check that

$$\boxed{e_{1/2}(t;\lambda) := E_{1/2}(-\lambda\,\sqrt{t}) = \mathrm{e}^{\lambda^2 t}\,\mathrm{erfc}(\lambda\sqrt{t}) \div \frac{1}{s^{1/2}\,(s^{1/2} + \lambda)}\,, \quad \lambda \in \mathbb{C}}\,. \tag{2.10}$$

Furthermore, *iff* $\lambda > 0$, our function $e_\alpha(t;\lambda)$ turns out to be *completely monotone* for $t > 0$, with certain *power-law* behaviours as $t \to 0^+$ and $t \to +\infty$, corresponding to an initial faster decay and a final slower decay with respect to the exponential decay of $\mathrm{e}^{-\lambda t}$, which is obtained in the limiting case $\alpha = 1$. In fact, $\underline{\text{for } \lambda > 0}$ one can prove

$$\boxed{e_\alpha(t;\lambda) := \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha(r;\lambda)\, dr \sim \begin{cases} 1 - \lambda\, \dfrac{t^\alpha}{\Gamma(\alpha+1)} & \text{as} \quad t \to 0^+\,, \\[2mm] \dfrac{1}{\lambda}\, \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} & \text{as} \quad t \to +\infty\,, \end{cases}} \tag{2.11}$$

with *spectral function*

$$\boxed{K_\alpha(r;\lambda) = \frac{1}{\pi}\, \frac{\lambda\, r^{\alpha-1}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\,\lambda\, r^\alpha\, \cos(\alpha\pi) + \lambda^2} \geq 0}\,. \tag{2.12}$$

In order to get some insight from the formulas (2.10-12) we find it useful to provide the reader with some instructive plots. For convenience we put $\lambda = 1$ and denote $e_\alpha(t) = e_\alpha(t;1)$, $K_\alpha(r) = K_\alpha(r;1)$.

In Fig. 2.1 we compare the plot of $e_{1/2}(t)$ (continuous line) with those of its asymptotic expressions for small and large t (dashed linen) and with the plot of $e_1(t) = \mathrm{e}^{-t}$ (dotted line). In Fig. 2.2 we compare the plots of $e_\alpha(t)$ for some values of α,in the interval $0 < \alpha \leq 1$,, namely $\alpha = 0.25, 0.50, 0.75, 1$. In Fig. 2.3 we compare the plots of the *spectral function* $K_\alpha(r)$ for some values of α in the interval $0 < \alpha < 1$,, namely $\alpha = 0.25, 0.50, 0.75, 0.90$.

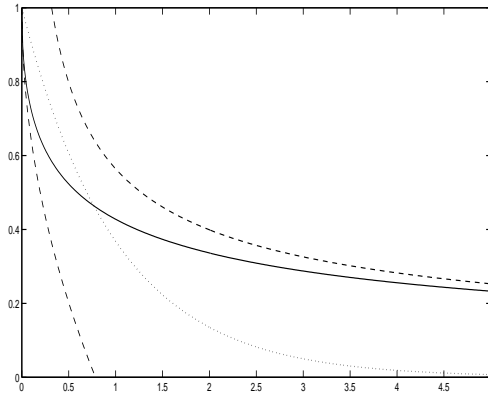**Fig. 2.1** – The *Mittag-Leffler function* of order $\alpha = 1/2$ compared with
its asymptotic representations (dashed lines) and the exponential (dotted line).
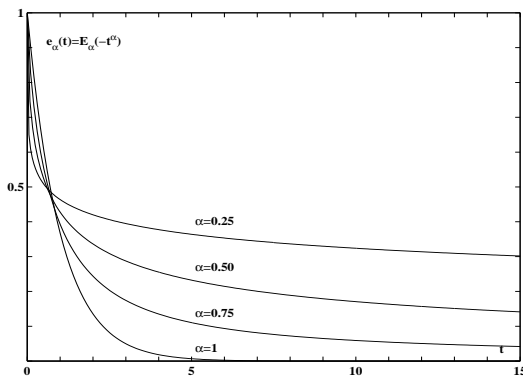

Fig. 2.2 – Plots of the *Mittag-Leffler function* $e_\alpha(t) = E_\alpha(-t^\alpha)$ for some values of $\alpha \in (0,1)$.
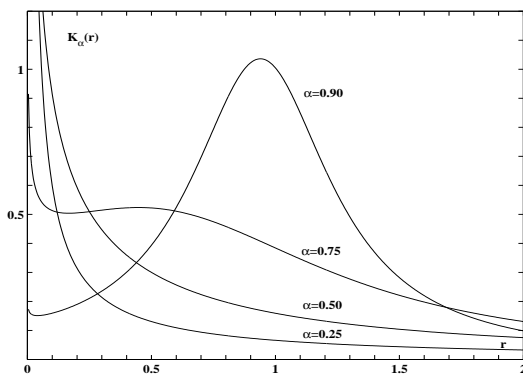

Fig. 2.3 – Plots of the *spectral function* $K_\alpha(r)$ for some values of $\alpha \in (0,1)$.

3. Abel Integral Equations via Fractional Calculus

After having solved the Abel equations by the standard technique of the Laplace transform, let us now interpret both the equations and their respective solutions in terms of Riemann-Liouville *fractional calculus*, which leads to a succinctness of notation and simplicity of formulation. We point out that, originally, Abel (1823, 1826) was able to solve his equation (2.1) by the tools of a primitive fractional calculus, before Liouville (1832, 1834, 1855, 1873) and Riemann (1847) put the basis of such a calculus, so named in their honour.

Let us first recall the basic definitions of *fractional integral J^α* and *fractional derivative D^α* of any order $\alpha > 0$ for a *causal* function $f(t)$. For more details we refer the reader to the author's lecture notes entitled *An Introduction to Fractional Calculus* and to the references there quoted.

We define the *fractional integral of order $\alpha > 0$* :

$$J^\alpha f(t) := \Phi_\alpha(t) * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau\,, \quad t > 0\,, \quad \alpha \in \mathbb{R}^+ . \quad (3.1)$$

For complementation we define $J^0 := I$ (Identity operator), *i.e.* we mean $J^0 f(t) = f(t)$. We note that for $\alpha = n \in \mathbb{N}$ the above definition reduces to the well known formula (usually attributed to Cauchy), that provides the $n-$fold primitive $f_n(t)$, (vanishing at $t = 0$ with its derivatives of order $1, 2, \ldots, n-1$) in terms a single integral of convolution type.

As the derivative of order n, D^n, turns out to be *left inverse* to the integral of order n, J^n, namely $D^n J^n = I$, so we expect that D^α is defined as left-inverse to J^α. For this purpose, introducing the positive integer m such that $m - 1 < \alpha \le m$, we define the *fractional derivative of order $\alpha > 0$* :

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau \right],$$
$$t > 0\,, \quad m - 1 < \alpha \le m\,, \quad m \in \mathbb{N}\,. \quad (3.2)$$

Using the semigroup property of the fractional integrals, *i.e.*

$$J^\alpha J^\beta = J^{\alpha+\beta}\,, \quad \alpha, \beta \ge 0\,, \quad (3.3)$$

it is easy to verify that $D^\alpha J^\alpha = I$. In fact, $D^\alpha J^\alpha = D^m J^{m-\alpha} J^\alpha = D^m J^m = I$. We recall that in general $J^\alpha D^\alpha \ne I$, as it occurs for the ordinary differentiation and integration operators.

It is useful to have in mind the effect of our operators J^α and D^α on the power functions t^γ with $\gamma > -1$, $t > 0$. We have, for $\alpha \ge 0$, the relations

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \quad t > 0\,, \gamma > -1 . \quad (3.4)$$

It is now easy to recognize that the Abel equation of the *first kind* (2.1) can be written as

$$\boxed{J^{\alpha}\,u(t) = f(t)}\,. \tag{3.5}$$

Consequently, applying the operator D^{α}, we obtain the solution as

$$\boxed{u(t) = D^{\alpha}\,f(t)}\,, \tag{3.6}$$

which is equivalent to (2.5a), if we put $m = 1$ in (3.2).

Let us now consider the Abel equation of the *second kind* (2.2). In terms of the fractional integral operator such equation reads

$$\boxed{(1 + \lambda\,J^{\alpha})\,u(t) = f(t)}\,, \tag{3.7}$$

and, consequently, can be formally solved as follows

$$\boxed{u(t) = (1 + \lambda J^{\alpha})^{-1}\,f(t) = \left(1 + \sum_{n=1}^{\infty}(-\lambda)^{n}\,J^{\alpha n}\right)\,f(t)}\,. \tag{3.8}$$

Noting by (3.1) that

$$J^{\alpha n}\,f(t) = \Phi_{\alpha n}(t)\,*\,f(t) = \frac{t_{+}^{\alpha n - 1}}{\Gamma(\alpha n)}\,*\,f(t)\,,$$

the formal solution reads

$$u(t) = f(t) + \left(\sum_{n=1}^{\infty}(-\lambda)^{n}\,\frac{t_{+}^{\alpha n - 1}}{\Gamma(\alpha n)}\right)\,*\,f(t)\,. \tag{3.9}$$

Recalling from the previous section the definition of the Mittag-Leffler type function,

$$\boxed{e_{\alpha}(t;\lambda) := E_{\alpha}(-\lambda\,t^{\alpha}) = \sum_{n=0}^{\infty}\frac{(-\lambda\,t^{\alpha})^{n}}{\Gamma(\alpha n + 1)}\,, \quad t > 0\,, \quad \alpha > 0\,, \quad \lambda \in \mathbb{C}}\,, \tag{3.10}$$

we note that

$$\sum_{n=1}^{\infty}(-\lambda)^{n}\,\frac{t_{+}^{\alpha n - 1}}{\Gamma(\alpha n)} = \frac{d}{dt}E_{\alpha}(-\lambda t^{\alpha}) = e'_{\alpha}(t;\lambda)\,, \quad t > 0\,. \tag{3.11}$$

Finally, from (3.9-11) the solution reads in the form (2.8c).

Of course the above formal proof, based on the series development of the operator $(1 + \lambda J^{\alpha})^{-1}$, can be made rigorous.

4. Inverse Problems for Heat Conduction

A field in which Abel integral equations or integral equations with more general weakly singular kernels are important is that of *inverse boundary value problems* in partial differential equations, in particular parabolic ones in which naturally the independent variable has the meaning of time. We are going to describe in detail the occurrence of Abel integral equations of first and of second kind in the problem of heating (or cooling) of a semi-infinite rod by influx (or efflux) of heat across the boundary into (or from) its interior.

Consider the *equation of heat flow*

$$\boxed{u_t - u_{xx} = 0\,, \quad u = u(x,t)\,, \quad x > 0\,, \quad t > 0}\,. \tag{4.1}$$

In this dimensionless equation $u = u(x,t)$ means temperature. Assume vanishing initial temperature, *i.e.* $u(x,0) = 0$ for $x > 0$ and given influx across the boundary $x = 0$ from $x < 0$ to $x > 0$,

$$-u_x(0,t) = p(t)\,. \tag{4.2}$$

Then, under appropriate regularity conditions, $u(x,t)$ can be shown to be given by the formula

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t - \tau}}\, \mathrm{e}^{-x^2/[4(t-\tau)]}\, d\tau\,, \quad x > 0\,, \quad t > 0\,. \tag{4.3}$$

We turn our special interest to the interior boundary temperature $\phi(t) := u(0^+, t)$, $t > 0$, which by (4.3) is represented as

$$\boxed{\phi(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t - \tau}}\, d\tau = J^{1/2}\, p(t)\,, \quad t > 0}\,. \tag{4.4}$$

We recognize (4.4) as an Abel integral equation of *first kind* for *determination of an unknown influx $p(t)$ if the interior boundary temperature $\phi(t)$ is given* by measurements, or intended to be achieved by controlling the influx. Its solution is therefore given by

$$\boxed{p(t) = D^{1/2}\, \phi(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\phi(\tau)}{\sqrt{t - \tau}}\, d\tau}\,, \tag{4.5}$$

where we have used the formula (3.2) for the fractional derivative with $m = 1$ and $\alpha = 1/2$.

It may be illuminating to consider the following special cases,

$$\boxed{(i)\ \ \phi(t) = t \implies p(t) = \frac{1}{2}\sqrt{\pi t}\,, \quad (ii)\ \ \phi(t) = 1 \implies p(t) = \frac{1}{\sqrt{\pi t}}}\,, \tag{4.6}$$

where we have used the formula in (3.4) for the fractional derivative of t^γ with $\gamma = 1$ and $\gamma = 0$. So, for linear increase of interior boundary temperature the required influx is continuous and increasing from 0 towards ∞ (with unbounded derivative at $t = 0^+$), whereas for instantaneous jump-like increase from 0 to 1 the required influx decreases from ∞ at $t = 0^+$ to 0 as $t \to \infty$.

We now modify our problem to obtain an Abel integral equation of *second kind.* Assume that the rod $x > 0$ is bordered at $x = 0$ by a bath of liquid in $x < 0$ with controlled exterior boundary temperature $u(0^-, t) := \psi(t)$.

Assuming Newton's radiation law we have an influx of heat from 0^- to 0^+ proportional to the difference of exterior and interior temperature,

$$p(t) = \lambda \left[\psi(t) - \phi(t)\right], \quad \lambda > 0.$$ (4.7)

Inserting (4.7) into (4.4) we obtain

$$\phi(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\psi(\tau) - \phi(\tau)}{\sqrt{t - \tau}} \, d\tau,$$

namely, in operator notation,

$$\boxed{\left(1 + \lambda J^{1/2}\right) \phi(t) = \lambda J^{1/2} \psi(t)}.$$ (4.8)

If we *now assume the exterior boundary temperature $\psi(t)$ as given and the evolution in time of the interior boundary temperature $\phi(t)$ as unknown*, then (4.23) is an Abel integral equation of *second kind* for determination of $\phi(t)$.

With $\alpha = 1/2$ the equation (4.8) is of the form (3.7), and by (3.8) its solution is

$$\boxed{\phi(t) = \lambda \left(1 + \lambda J^{1/2}\right)^{-1} J^{1/2} \psi(t) = -\sum_{m=0}^{\infty} (-\lambda)^{m+1} J^{(m+1)/2} \psi(t)}.$$ (4.9)

Let us investigate the very special case of constant exterior boundary temperature,

$$\boxed{\psi(t) = 1}.$$ (4.10)

Then, by using the formula in (3.4) for the fractional integral of t^γ with $\gamma = 0$, we obtain

$$J^{(m+1)/2} \psi(t) = \frac{t^{(m+1)/2}}{\Gamma\left[(m+1)/2 + 1\right]},$$

hence

$$\phi(t) = -\sum_{m=0}^{\infty} (-\lambda)^{m+1} \frac{t^{(m+1)/2}}{\Gamma\left[(m+1)/2 + 1\right]} = 1 - \sum_{n=0}^{\infty} (-\lambda)^n \frac{t^{n/2}}{\Gamma\left(n/2 + 1\right)},$$

so that

$$\boxed{\phi(t) = 1 - E_{1/2}\left(-\lambda t^{1/2}\right) = 1 - e_{1/2}(t; \lambda)}.$$ (4.11)

Observe that $\phi(t)$ is a *creep function*, increasing strictly monotonically from 0 towards 1 as t runs from 0 to ∞.

5. The Tautochrone Problem

Niels Henrik ABEL was led to his famous equation by the mechanical problem of the TAUTOCHRONE, that is, by the problem of determining a curve in the vertical plane, such that the time required for a particle to slide down the curve (without friction) to its lowest point is equal to a given function of its initial height. As a special case Abel discussed the problem of the ISOCHRONE, in which it is required that the time of sliding down is *independent* of the initial height.

Let us fix the lowest point of the curve at the origin and position the curve in the positive quadrant of the plane, denoting by (X, Y) the initial point and (x, y) any point intermediate between $(0, 0)$ and (X, Y), see Fig. 5.1.
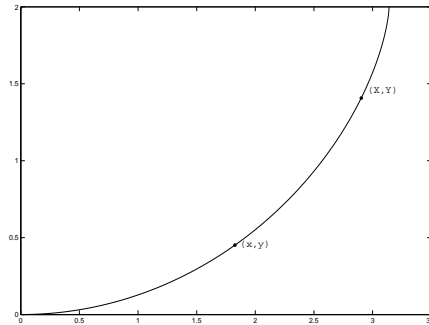

Fig. 5.1 The coordinate system for the tautochrone.

Assuming no frictional loses, we obtain from the conservation of energy

$$\boxed{\frac{m}{2} \left(\frac{ds}{dt}\right)^2 = mg\,(Y - y)}\,, \tag{5.1}$$

where s is the arc length along the curve measured from the origin, m is the mass of the particle, g the gravitational acceleration, and t the time. Thus, since $ds/dt < 0$,

$$ds = -\sqrt{2g\,(Y - y)} = -|v|\,dt\,, \tag{5.2}$$

where v denotes the scalar velocity. Then, the total time of descent can be computed by the following integral

$$T = T(Y) = \int_0^{s(Y)} \frac{ds}{|v|}\,, \tag{5.3}$$

if we write $s = s[y(t)]$ to indicate the dependence of s on the height y (which itself depends on time). We thus obtain

$$\boxed{T(Y) = \frac{1}{\sqrt{2g}} \int_0^Y \frac{(ds/dy)}{\sqrt{Y - y}}\,dy}\,. \tag{5.4}$$

Putting provisionally

$$u(y) = \frac{ds}{dy}, \quad f(y) = \sqrt{\frac{2g}{\pi}}\, T(y), \tag{5.5}$$

we can re-write (5.4) in terms of *fractional calculus, i.e.*

$$\frac{1}{\Gamma(1/2)} \int_0^Y \frac{u(y)}{\sqrt{Y-y}}\, dy = J^{1/2}\, u(Y) = f(Y). \tag{5.6}$$

The solution is therefore

$$u(Y) = D^{1/2}\left[f\left(Y\right)\right] = \frac{1}{\Gamma(1/2)} \frac{d}{dY} \int_0^Y \frac{f(y)}{\sqrt{Y-y}}\, dy. \tag{5.7}$$

Now, using (5.5) and (5.7), we can express $s(Y)$ in terms of $T(Y)$ through the following integral relation, which provides the solution of the original problem

$$\boxed{s(Y) = \frac{\sqrt{2g}}{\pi} \int_0^Y \frac{T(y)}{\sqrt{Y-y}}\, dy}. \tag{5.8}$$

Let us now consider two noteworthy particular cases of $T(y)$ provided by the problems of the *free fall* and of the *isochrone*, respectively.

For the problem of the *free fall* it is $y = g\,[T(y)]^2/2$, and therefore

$$\boxed{T(y) = \sqrt{\frac{2y}{g}} \implies s(Y) = \frac{2}{\pi} \int_0^Y \frac{\sqrt{y}}{\sqrt{Y-y}}\, dy = Y}, \tag{5.9}$$

as expected. The result can be obtained by putting $y = Y\,\sin^2\phi$, $0 \le \phi \le \pi/2$.

For the problem of the *isochrone* it is easy to show

$$\boxed{T(y) = T = const \implies s(Y) = \frac{\sqrt{2g}\, T}{\pi} \int_0^Y \frac{dy}{\sqrt{Y-y}} = 2\,\sqrt{2R}\, Y^{1/2}}, \tag{5.10}$$

where we have put

$$\boxed{R = g\,\frac{T^2}{\pi^2}}. \tag{5.11}$$

Let us show that the *isochrone* turns out to be the *cycloid* generated by a circle of radius R, which rolls without slipping under the line $y = 2R$.

For a *resumé* on the *cycloid* we refer to Appendix 1. Following the notation of the Appendix the parametric equations of the cycloid turn out to be

$$s = s(\theta) = 4R \sin \theta/2 \iff \begin{cases} x = R\,(\theta + \sin \theta) \\ y = R\,(1 - \cos \theta) \end{cases} - \pi \le \theta \le +\pi\,, \qquad (5.12)$$

where θ is the *rotation angle*. The cycloid requires that the angle ψ between the tangent to the curve and the positive x-axis be one half of the rotation angle θ, *i.e.*

$$\mathrm{tg}\,\psi = \frac{dy}{dx}\,, \qquad \psi = \frac{\theta}{2}\,. \qquad (5.13)$$

Consequently the parametric equations of the cycloid can be expressed in terms of the angle ψ as follows

$$s = s(\psi) = 4R \sin \psi \iff \begin{cases} x = R\,(2\,\psi + \sin 2\,\psi) \\ y = 2R\,\sin^2 \psi \end{cases} - \pi/2 \le \psi \le +\pi/2\,. \qquad (5.14)$$

In order to prove that the *isochrone* is just the above *cycloid* we must show that

$$s(y) = 2\,\sqrt{2R}\,y^{1/2}\,, \qquad (ds/dy) = \sqrt{2R}\,y^{-1/2}\,, \qquad (5.15)$$

derived from (5.10), lead to the equations in (5.14). While the equations for $s = s(\psi)$ and $y = y(\psi)$ are easily verified, that for $x(\psi)$ requires some computations. In fact from $(dx)^2 + (dy)^2 = (ds)^2$, $u(y) = ds/dy$ and $y = 2R\,\sin^2 \psi$, we have

$$x = \int \sqrt{u^2(y) - 1}\,dy = \int \sqrt{2R/y - 1}\,dy = 4R \int \sqrt{1/\sin^2 \psi - 1}\,\sin \psi\,\cos \psi\,d\psi$$

$$= 4R \int \cos^2 \psi\,d\psi = 2R \int (1 + \cos 2\psi)\,d\psi = R\,(2\psi + \sin 2\psi)\,.$$

Now we are in position to compute the (constant) period of a *cycloidal pendulum*, formerly derived by Huygens (1673). In fact, multiplying by four the descent time in (5.11) in order to allow a complete oscillation, we obtain

$$T_H = 4T = 2\pi\,\sqrt{\frac{4R}{g}}\,. \qquad (5.16)$$

It is instructive to compare this period with that of the ordinary *circular pendulum* (of length L) for small oscillations, formerly derived by Galileo (1581),

$$T_G \approx 2\pi\,\sqrt{\frac{L}{g}}\,. \qquad (5.17)$$

In order to get the identity between the two periods we must take $L = 4R$ and assume $\sin \alpha \approx \alpha$, as described in Appendix 2, where the equations of motion for both the *cycloidal pendulum* and the *circular pendulum* are derived.
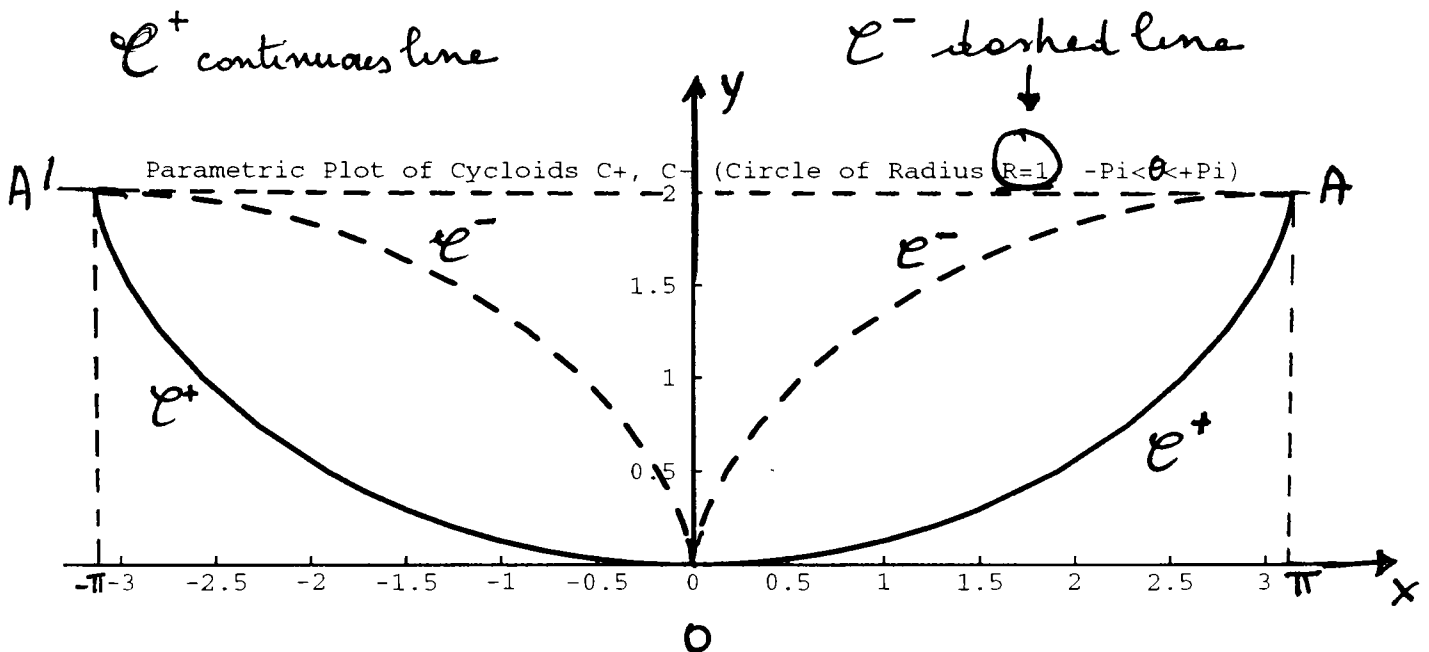
APPENDIX 1: THE CYCLOID

$$\mathcal{C}^{\pm} \equiv \begin{cases} x = R(\theta \pm \sin\theta) \\ y = R(1 - \cos\theta) \end{cases} \qquad -\pi \le \theta \le \pi \quad \triangle$$

$$\begin{cases} dx = R(1 \pm \cos\theta)\, d\theta \\ dy = R \sin\theta\, d\theta \end{cases} \Rightarrow \boxed{ds = 2R \cos\theta/2\, d\theta \Rightarrow s(\theta) = 4R \sin\theta/2}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1 \pm \cos\theta} = \begin{cases} tg\,\theta/2\ (\mathcal{C}^+) \\ ctg\,\theta/2\ (\mathcal{C}^-) \end{cases} \quad ; \quad \frac{d\tilde{y}}{dx^2} = \pm \frac{1}{(1\pm\cos\theta)^2} = \begin{cases} \dfrac{1}{4\cos^2\theta/2} > 0\ (\mathcal{C}^+) \\[2mm] \dfrac{-1}{4\sin^2\theta/2} < 0\ (\mathcal{C}^-) \end{cases}$$

$$\text{Let us recall} \begin{cases} \cos^2\alpha - \sin^2\alpha = \cos 2\alpha \\ 2\cos\alpha\,\sin\alpha = \sin 2\alpha \end{cases} \Rightarrow \begin{cases} \sin^2\alpha = (1 - \cos 2\alpha)/2 \\ \cos^2\alpha = (1 + \cos 2\alpha)/2 \end{cases}$$

$\mathcal{C}^+$ continuous line $\mathcal{C}^-$ dashed line

Parametric Plot of Cycloids C+, C- (Circle of Radius R=1 -Pi<θ<+Pi)

$$A' \Leftrightarrow \theta = -\pi \begin{cases} x = -\pi R \\ y = 2R \end{cases} ; \quad 0 \Leftrightarrow \theta = 0 \begin{cases} x = 0 \\ y = 0 \end{cases} ; \quad A \Leftrightarrow \theta = \pi \begin{cases} x = \pi R \\ y = 2R \end{cases}$$

THE CYCLOID IS THE LOCUS OF A POINT ON THE CIRCUMFERENCE
OF A CIRCLE (OF RADIUS R) WHICH <u>ROLLS WITHOUT SLIPPING</u>
ON A STRIGHT LINE ! $\triangle$
FOR $\mathcal{C}^-$ the circle rolls <u>over</u> the line $y = 0$
FOR $\mathcal{C}^+$ the circle rolls <u>under</u> the line $y = 2R$

APPENDIX 2: THE CYCLOIDAL PENDULUM

EQUATION OF MOTION FOR A PARTICLE MASS CONSTRAINED
TO MOVE IN A VERTICAL CURVE, WITHOUT FRICTION

$$F_t(s) = m \frac{d^2 s}{dt^2} \quad ; \quad F_t(s) = -mg \cos\phi \quad ; \quad \cos\phi = \frac{dy}{ds}$$

CYCLOID (generated by a circle of radius R)

$$\begin{cases} x = R(\theta + \sin\theta) \\ y = R(1-\cos\theta) \end{cases} \quad -\pi \le \theta \le \pi$$

$$\begin{cases} dx = R(1+\cos\theta)d\theta \\ dy = R\sin\theta\, d\theta \end{cases} \Rightarrow ds = 2R\cos\frac{\theta}{2}d\theta$$
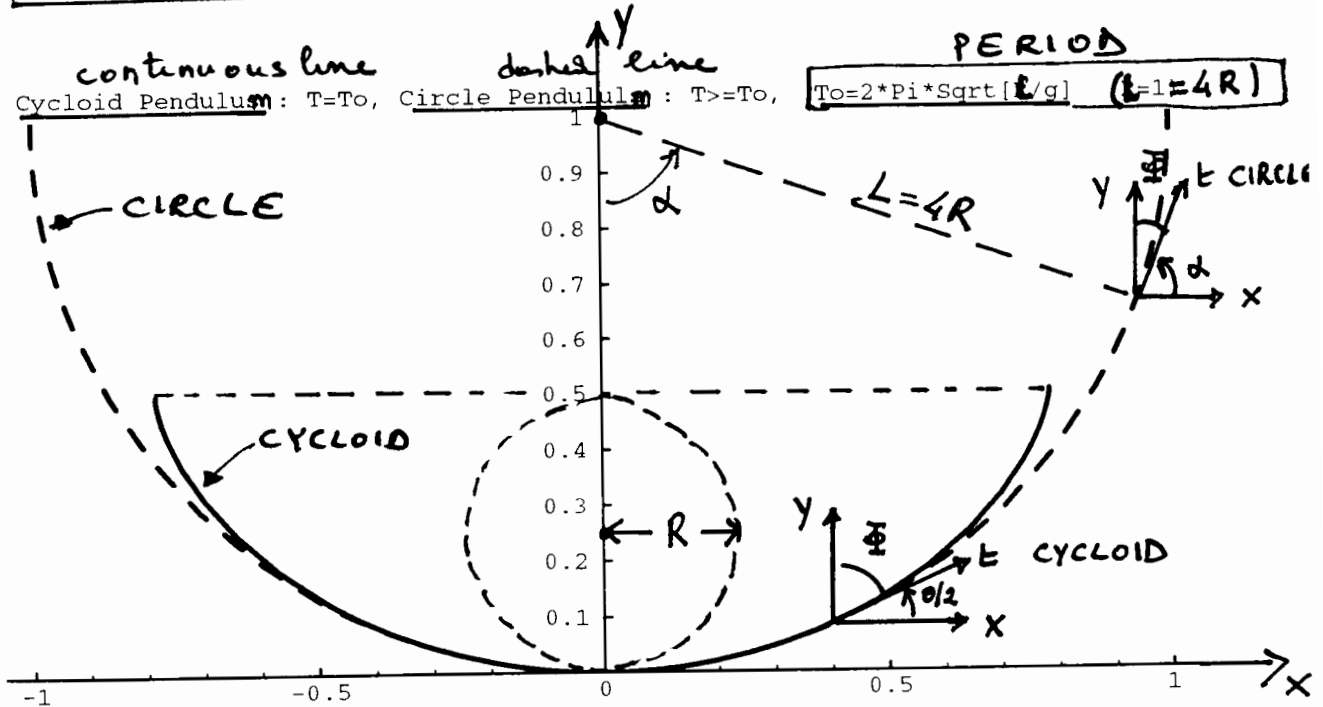
$$\boxed{S = S(\theta) = 4R \sin\theta/2}$$

$(*)$

CIRCLE (of radius L)

$$\begin{cases} x = L\sin\alpha \\ y = L(1-\cos\alpha) \end{cases} \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$$

$$\begin{cases} dx = L\cos\alpha\, d\alpha \\ dy = L\sin\alpha\, d\alpha \end{cases} \quad dS = L\, d\alpha$$

$$\boxed{S = S(\alpha) = L\alpha}$$

continuous line   dashed line

Cycloid Pendulum: T=To, Circle Pendulum: T>=To,

PERIOD

$$\boxed{T_o = 2\pi \cdot \text{Sqrt}[L/g]} \quad \boxed{L=1=4R}$$

CIRCLE

$L = 4R$

CYCLOID

$\leftarrow R \rightarrow$

$$\cos\phi = \frac{dy}{ds} = \frac{\sin\theta}{2\cos\theta/2} = \sin\theta/2$$

CYCLOID

$$\frac{d^2s}{dt^2} + g\sin\frac{\theta}{2} = 0 \Rightarrow (*)$$

$$\boxed{\frac{d^2s}{dt^2} + \frac{g}{4R}s = 0}$$

$(**)$

HARMONIC MOTION $\forall s = s(\theta)$

⚠

$$\boxed{T = 2\pi\sqrt{\frac{4R}{g}}}$$

HUYGENS

$$\cos\phi = \frac{dy}{ds} = \sin\alpha$$

CIRCLE

$$\frac{d^2s}{dt^2} + g\sin\alpha = 0 \Rightarrow (*)$$

$$\boxed{\frac{d^2\alpha}{dt^2} + \frac{g}{L}\sin\alpha = 0}$$

$(***)$

HARMONIC MOTION ONLY for small α

⚠

$$\boxed{T \simeq 2\pi\sqrt{\frac{L}{g}}}$$

GALILEO

FISICA MATEMATICA: CYC.curf.ps

S. XII. 93

HUYGENS (1673)
EULER (1736)

(1581)

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