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Salvatore Pincherle: the pioneer of the Mellin–Barnes integrals

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Abstract

The 1888 paper by Salvatore Pincherle (Professor of Mathematics at the University of Bologna) on generalized hypergeometric functions is revisited. We point out the pioneering contribution of the Italian mathematician towards the Mellin–Barnes integrals based on the duality principle between linear differential equations and linear difference equation with rational coefficients. By extending the original arguments used by Pincherle, we also show how to formally derive the linear differential equation and the Mellin–Barnes integral representation of the Meijer G functions.

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1. Preface

In Vol. 1, p. 49 of *Higher Transcendental Functions* of the Bateman Project [5] we read "Of all integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin–Barnes integrals. Such integrals were first introduced by Pincherle, in 1888 [21]; their theory has been developed in 1910 by Mellin (where there are references to earlier work) [17] and they were used for a complete integration of the hypergeometric differential equation by Barnes [2]".

In the classical treatise on Bessel functions by Watson [27, p. 190], we read "By using integrals of a type introduced by Pincherle and Mellin, Barnes has obtained representations of Bessel functions..."

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Salvatore Pincherle (1853–1936) was Professor of Mathematics at the University of Bologna from 1880 to 1928. He retired from the University just after the International Congress of Mathematicians that he had organized in Bologna, following the invitation received at the previous Congress held in Toronto in 1924. He wrote several treatises and lecture notes on Algebra, Geometry, Real and Complex Analysis. His main book related to his scientific activity is entitled "Le Operazioni Distributive e loro Applicazioni all'Analisi"; it was written in collaboration with his assistant, Dr. Ugo Amaldi, and was published in 1901 by Zanichelli, Bologna. Pincherle can be considered one of the most prominent founders of the Functional Analysis, as pointed out by J. Hadamard in his review lecture "Le développement et le rôle scientifique du Calcul fonctionnel", given at the Congress of Bologna (1928). A description of Pincherle's scientific works requested from him by Mittag–Leffler, who was the Editor of Acta Mathematica, appeared (in French) in 1925 on this prestigious journal [22]. A collection of selected papers (38 from 247 notes plus 24 treatises) was edited by Unione Matematica Italiana (UMI) on the occasion of the centenary of his birth, and published by Cremonese, Roma 1954. Note that Pincherle was the first President of UMI, from 1922 to 1936.

Here we point out that the 1888 paper (in Italian) of Pincherle on the *Generalized Hypergeometric Functions* led him to introduce the afterwards named Mellin–Barnes integral to represent the solution of a generalized hypergeometric differential equation investigated by Goursat in 1883. Pincherle's priority was explicitly recognized by Mellin and Barnes themselves, as reported below.

In 1907 Barnes, see p. 63 in [1], wrote: "The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle, whose suggestive paper was the starting point of the investigations of Mellin (1895) though the type of contour and its use can be traced back to Riemann". In 1910 Mellin, see p. 326ff in [17], devoted a section (Section 10: Proof of Theorems of Pincherle) to revisit the original work of Pincherle; in particular, he wrote "Before we are going to prove this theorem, which is a special case of a more general theorem of Pincherle, we want to describe more closely the lines L over which the integration preferably is to be carried out". [free translation from German].

The Mellin–Barnes integrals are the essential tools for treating the two classes of higher transcendental functions known as G and H functions, introduced by Meijer (1946) [13] and Fox (1961) [6], respectively, so Pincherle can be considered their precursor. For an exhaustive treatment of the Mellin–Barnes integrals, we refer to the recent monograph by Paris and Kaminski [19].

The purpose of our paper is to let know the community of scientists interested in special functions the pioneering 1888 work by Pincherle, that, in the author's intention, was devoted to compare two different generalizations of the Gauss hypergeometric function due to Pochhammer and to Goursat. Incidentally, for a particular case of the Goursat function, Pincherle used an integral representation in the complex plane that in future was adopted by Mellin and Barnes for their treatment of the *generalized hypergeometric functions* known as ${}_pF_q(z)$. We also intend to show, in the original part of our paper, that, by extending the original arguments by Pincherle, we are able to provide the Mellin–Barnes integral representation of the transcendental functions introduced by Meijer (the so-called G functions).

The paper is divided as follows. In Section 2, we report the major statements and results of the 1888 paper by Pincherle. In Section 3, we show how it is possible to originate from these results the Meijer G functions by a proper generalization of Pincherle's method. Finally, Section 4 is devoted to the conclusions. We find it convenient to reserve an appendix for recalling some basic notions for the generalized hypergeometric functions and the Meijer G functions.

2. The Pochhammer and Goursat generalized hypergeometric functions via Pincherle's arguments

The 1888 paper by Pincherle is based on what he called "duality principle", which relates linear differential equations with rational coefficients to linear difference equations with rational coefficients. Let us remind that the sentence "rational coefficients" means that the coefficients are in general rational functions (i.e. ratio between two polynomials) of the independent variable and, in particular, polynomials. By using this principle with polynomial coefficients Pincherle showed that two generalized hypergeometric functions proposed and investigated, respectively, by Pochhammer (1870), see [23], and by Goursat (1883), see [7,8], can be obtained and related to each other. [1]

The generalized hypergeometric functions introduced by Pochhammer and Goursat considered by Pincherle are solutions of linear differential equations of order n with polynomial coefficients, that we report in the appendix. As a matter of fact, the duality principle states the correspondence between a linear *ordinary differential equation* (*ODE*) and a linear *finite difference equation* (*FDE*). The coefficients of both equations are assumed to be *rational* functions, in particular *polynomials*. In his analysis [21], Pincherle considered the correspondence between the following equations,

$$\sum_{h=0}^{m}(a_{h0}+a_{h1}\mathrm{e}^{-t}+a_{h2}\mathrm{e}^{-2t}+\cdots+a_{hp}\mathrm{e}^{-pt})\psi^{(h)}(t)=0, \tag{2.1}$$

$$\sum_{k=0}^{p}[a_{0k}+a_{1k}(x+k)+a_{2k}(x+k)^2+\cdots+a_{mk}(x+k)^m]f(x+k)=0, \tag{2.2}$$

where $\psi(t)$ and $f(x)$ are analytic functions. These functions are required to be related to each other through a Laplace-type transformation $\psi(t)\leftrightarrow f(x)$ defined by the formulas

$$\text{(a)}\ f(x)=\int_l \mathrm{e}^{-xt}\psi(t)\,\mathrm{d}t,\quad \text{(b)}\ \psi(t)=\frac{1}{2\pi\mathrm{i}}\int_L \mathrm{e}^{+xt}f(x)\,\mathrm{d}x, \tag{2.3}$$

where l and L are appropriate integration paths in the complex t and x plane, respectively.

The singular points of the ODE are the roots of the polynomial

$$a_{m0}+a_{m1}z+a_{m2}z^2+\cdots+a_{mp}z^p=0 \tag{2.4}$$

whereas the singular points of the FDE are the roots of the polynomial

$$a_{00}+a_{10}z+a_{20}z^2+\cdots+a_{m0}z^m=0. \tag{2.5}$$

For the details of the above correspondence Pincherle refers to the 1885 fundamental paper by Poincaré [24] [2] and his own 1886 note [20]. Here, we limit ourselves to point out what can be

[1] In fact, translating from Italian, the author so writes in introducing his paper: "It is known that to any linear differential equation with rational coefficients one may let correspond a linear difference equation with rational coefficients. In other words, if the former equation is given, one can immediately write the latter one and vice versa; furthermore, from the integral of the one, the integral of the latter can be easily deduced. This *relationship* appears to be originated by a sort of *duality principle* of which, in this note, I want to treat an application concerning generalized hypergeometric functions".

[2] For an account of Poincaré's theorem upon which Pincherle based his analysis the interested reader can consult the recent book by Elaydi [4, pp. 320–323].

easily seen from a formal comparison between the ODE (2.1) and the FDE (2.2). We recognize that the degree p of the coefficients in e^{-t} of the ODE provides the order of the FDE, and that the order m of the ODE gives the degree in x of the coefficients of the FDE. Vice versa, the degree m of the coefficients of the FDE provides the order of the ODE, and the order p of the FDE gives the degree in e^{-t} of the coefficients of the ODE.

Pincherle's intention was to apply the above duality principle in order to compare the generalized hypergeometric function introduced by Pochhammer and governed by (A.7) with that by Goursat governed by (A.6). Using his words, he proved that the family of the Pochhammer functions (of arbitrary order p) originates from a linear FDE (of order p) whose coefficients are polynomials of the first degree in x, and that the family of the Goursat functions (of arbitrary order m) originates from a linear ODE (of order m) whose coefficients are polynomials of the first degree in $x = e^{-t}$. As a consequence of the duality principle there is a mutual correspondence between the properties of the functions belonging to one family and to the other.

For the Pochhammer function he started from the ODE of the first order

$$\begin{aligned}(a_{00} + a_{01}e^{-t} + a_{02}e^{-2t} + \cdots + a_{0p}e^{-pt})\psi(t)\\ + (a_{10} + a_{11}e^{-t} + a_{12}e^{-2t} + \cdots + a_{1p}e^{-pt})\psi^{(1)}(t) = 0,\end{aligned} \tag{2.6}$$

to be put in correspondence with the FDE

$$\begin{aligned}(a_{00} + a_{10}x)f(x) + [a_{01} + a_{11}(x+1)]f(x+1) + [a_{02} + a_{12}(x+2)]f(x+2)\\ + \cdots + [a_{0p} + a_{1p}(x+p)]f(x+p) = 0.\end{aligned} \tag{2.7}$$

In this case Pincherle was able to show that the solution $f(x)$ of the FDE (2.7), obtained through the formula (a) in (2.3), depends on p parameters, whose logarithms are the singular points of the ODE (2.6). With respect to each of these parameters, $f(x)$ satisfies a linear ODE of the Pochhammer type of order p.

For the Goursat function he started from a FDE of the first order

$$\begin{aligned}[a_{00} + a_{10}x + a_{20}x^2 + \cdots + a_{m0}x^m]f(x)\\ + [a_{01} + a_{11}(x+1) + a_{21}(x+1)^2 + \cdots + a_{m1}(x+1)^m]f(x+1) = 0,\end{aligned} \tag{2.8}$$

to be put in correspondence to the linear ODE of order m

$$\begin{aligned}(a_{00} + a_{01}e^{-t})\psi(t) + (a_{10} + a_{11}e^{-t})\psi^{(1)}(t) + (a_{20} + a_{21}e^{-t})\psi^{(2)}(t)\\ + \cdots + (a_{m0} + a_{m1}e^{-t})\psi^{(m)}(t) = 0.\end{aligned} \tag{2.9}$$

Using a result of Mellin, see [14,15], Pincherle wrote the solution of the FDE (2.8) as

$$f(x) = c^x \prod_{v=1}^{m} \frac{\Gamma(x-\rho_v)}{\Gamma(x-\sigma_v)}, \tag{2.10}$$

where the ρ_v's and σ_v's are, respectively, the roots of the algebraic equations

$$a_{00} + a_{10}x + \cdots + a_{m0}x^m = a_{m0}\prod_{v=1}^{m}(x-\rho_v) = 0,$$

$$a_{01} + a_{11}(x+1) + \cdots + a_{m1}(x+1)^m = a_{m1}\prod_{v=1}^{m}(x-\sigma_v) = 0 \tag{2.11}$$

and c is a constant. If a_{m0}, a_{m1} are both different from zero, we can assume $c = -a_{m0}/a_{m1}$.

Pincherle showed that, by setting $z = c\mathrm{e}^t$, the ODE of order m (2.9) is nothing but the Goursat differential equation (A.6).

Furthermore, in the special case $a_{m1} = 0$, he gave the following relevant formula for the solution

$$\psi(t) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{\Gamma(x-\rho_1)\Gamma(x-\rho_2)\cdots\Gamma(x-\rho_m)}{\Gamma(x-\sigma_1)\Gamma(x-\sigma_2)\cdots\Gamma(x-\sigma_{m-1})}\,\mathrm{e}^{xt}\,\mathrm{d}x, \tag{2.12}$$

where $a > \Re\{\rho_1, \rho_2, \ldots, \rho_m\}$. We recognize in (2.12) the first example in the literature of the (afterwards named) Mellin–Barnes integral. The convergence of the integral was proved by Pincherle by using his asymptotic formula for $\Gamma(a+\mathrm{i}\eta)$ as $\eta \to \pm\infty$.[3] So, for a solution of a particular case of the Goursat equation, Pincherle provided an integral representation that later was adopted by Mellin and Barnes for their treatment of the generalized hypergeometric functions ${}_pF_q(z)$. Since then, the merits of Mellin and Barnes were so well recognized that their names were attached to the integrals of this type; on the other hand, after the 1888 paper (written in Italian), Pincherle did not pursue on this topic, so his name was no longer related to these integrals and, in this respect, his 1888 paper was practically ignored.

3. The Meijer transcendental function via Pincherle's arguments

In more recent times other families of higher transcendental functions have been introduced to generalize the hypergeometric function based on their representation by Mellin–Barnes type integrals. We especially refer to the so-called G and H functions, briefly recalled in the appendix.

In this section (the original part of our paper), we show that by extending the original arguments by Pincherle based on the duality principle we are able to provide the differential equation and the Mellin–Barnes integral representation of the G functions. However, we note that these arguments, being based on equations with rational coefficients, do not allow us to treat the Fox H functions, since for them an ordinary differential equation cannot be found in the general case.

Our starting point is still the "duality principle" that involves a FDE *of the first order as in Pincherle's approach for the Goursat function*, but, at variance of Eq. (2.8), we now allow that the degree of the two polynomial coefficients are not necessarily equal. Setting p, q the degrees of these

[3] We also note the priority of Pincherle in obtaining this asymptotic formula, as outlined by Mellin, see e.g. [16, pp. 330–331], and [17, p. 309]. In his 1925 "Notices sur les travaux" [22, p. 56, Section 16] Pincherle wrote "Une expression asymptotique de $\Gamma(x)$ pour $x \to \infty$ dans le sens imaginaire qui se trouve dans [21] a été attribuée à d'autres auteurs, mais Mellin m'en a récemment révendiqué la priorité". This formula is fundamental to investigate the convergence of the Mellin–Barnes integrals, as one can recognize from the detailed analysis by Dixon and Ferrar [3], see also [19].

coefficients, our FDE reads

$$[a_{00}+a_{10}x+a_{20}x^2+\cdots+a_{p0}x^p]f(x) + [a_{01}+a_{11}(x+1)+a_{21}(x+1)^2+\cdots+a_{q1}(x+1)^q]f(x+1)=0. \tag{3.1}$$

We can prove after some algebra that the associated ODE turns out to be independent of the order relation between p and q and reads

$$\sum_{h=0}^{p} a_{h0}\psi^{(h)}(t)+\mathrm{e}^{-t}\sum_{h=0}^{q} a_{h1}\psi^{(h)}(t)=0. \tag{3.2}$$

As we have learnt from Pincherle's analysis, the solution $\psi(t)$ of ODE (3.2) can be expressed in terms of the solution $f(x)$ of FDE (3.1), according to the integral representation (b) in Eq. (2.3).

Now, in view of Mellin's results used by Pincherle (see also Milne–Thomson [18, Section 11.2, p. 327]), we can write the solution of (3.1) in terms of products and fractions of Γ functions. Denoting by ρ_j $(j=0,1,\ldots,p)$ and σ_k $(k=0,1,\ldots,q)$ the roots of the algebraic equations

$$a_{00}+a_{10}x+\cdots+a_{p0}x^p=a_{p0}\prod_{j=1}^{p}(x-\rho_j)=0,$$

$$a_{01}+a_{11}(x+1)+\cdots+a_{q1}(x+1)^q=a_{q1}\prod_{k=1}^{q}(x-\sigma_k)=0 \tag{3.3}$$

we can write the required solution as

$$f(x)=c^x\frac{\prod_{j=1}^{p}\Gamma(x-\rho_j)}{\prod_{k=1}^{q}\Gamma(x-\sigma_k)},\quad c=-\frac{a_{p0}}{a_{q1}}. \tag{3.4}$$

We note, by using the known properties of the Gamma function, that Eq. (3.4) can be re-written in the following alternative form:

$$f(x)=c^x\frac{\prod_{k=1}^{q}\Gamma(1+\sigma_k-x)}{\prod_{j=1}^{p}\Gamma(1+\rho_j-x)},\quad c=(-1)^{p-q+1}\frac{a_{p0}}{a_{q1}}. \tag{3.5}$$

Furthermore, introducing the integers m,n such that $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$, we can combine the previous formulas (3.4)–(3.5) and obtain the alternative form

$$f(x)=c^x\frac{\prod_{j=1}^{n}\Gamma(x-\rho_j)\prod_{k=1}^{m}\Gamma(1+\sigma_k-x)}{\prod_{j=n+1}^{p}\Gamma(1+\rho_j-x)\prod_{k=m+1}^{q}\Gamma(x-\sigma_k)} \tag{3.6}$$

with

$$c=(-1)^{m+n-p+1}\frac{a_{p0}}{a_{q1}}. \tag{3.7}$$

We note that Eq. (3.6) reduces to the Pincherle expression (2.10) by setting $\{n=p=q, m=0\}$, and to Eqs. (3.4), (3.5) by setting $\{n=p, m=0\}$, $\{n=0, m=q\}$, respectively. By adopting the form (3.6)–(3.7), we have the most general expression for $f(x)$ which in its turn allows us to arrive at the most general solution $\psi(t)$ of the corresponding ODE (3.2) in the form

$$\psi(t) = \frac{1}{2\pi \mathrm{i}} \int_L c^x \frac{\prod_{j=1}^{n} \Gamma(x-\rho_j) \prod_{k=1}^{m} \Gamma(1+\sigma_k - x)}{\prod_{j=n+1}^{p} \Gamma(1+\rho_j - x) \prod_{k=m+1}^{q} \Gamma(x-\sigma_k)} \mathrm{e}^{xt}\, \mathrm{d}x, \tag{3.8}$$

where L is an appropriate integration path in the complex x plane.

Now, starting from (3.2) and (3.7)–(3.8) it is not difficult to arrive at the general G function namely at its ODE and at its Mellin–Barnes integral representation, both given in the appendix. For this purpose, we need only to carry out some algebraic manipulations and obvious transformations of variables.

We first note that using (3.3) ODE (3.2) reads

$$\left[a_{p0} \prod_{j=1}^{p} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \rho_j \right) + a_{q1} \mathrm{e}^{-t} \prod_{k=1}^{q} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \sigma_k - 1 \right) \right] \psi(t) = 0. \tag{3.9}$$

Then, putting

$$z = c\mathrm{e}^{t}, \quad u(z) = \psi[t(z)], \quad a_j = 1 + \rho_j, \quad b_k = 1 + \sigma_k \tag{3.10}$$

and using (3.7), we get from (3.9)

$$\left[(-1)^{p-m-n} z \prod_{j=1}^{p} \left(z \frac{\mathrm{d}}{\mathrm{d}z} - a_j + 1 \right) - \prod_{k=1}^{q} \left(z \frac{\mathrm{d}}{\mathrm{d}z} - b_k \right) \right] u(z) = 0, \tag{3.11}$$

which is just the ODE satisfied by the Meijer G function of orders m, n, p, q, see (A.10). Of course, at least formally, the Mellin–Barnes integral representation of the G function (A.8)–(A.9) is recovered as well and reads (setting $s = x$)

$$u(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\prod_{k=1}^{m} \Gamma(b_k - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, \mathrm{d}s. \tag{3.12}$$

4. Conclusions

We have revisited the 1888 paper (in Italian) by Pincherle on generalized hypergeometric functions, based on the duality principle between linear differential equations and linear difference equation with rational coefficients. We have pointed out the pioneering contribution of the Italian mathematician towards the afterwards named Mellin–Barnes integral representation that he was able to provide for a special case of a generalized hypergeometric function introduced by Goursat in 1883. By extending his original arguments we have shown how to formally derive the ordinary differential equation and the Mellin–Barnes integral representation of the G functions introduced by Meijer in 1936–1946. So, in principle, Pincherle could have introduced the G functions much before Meijer if he had intended to pursue his original arguments in this direction. Finally, we like to point out that the so-called Mellin–Barnes integrals are an efficient tool to deal with the higher transcendental functions. In fact, for a pure mathematics view point they facilitate the representation of these functions (as formerly indicated by Pincherle), and for an applied mathematics view point they can be successfully adopted to compute the same functions. In this respect we refer to the recent paper by Mainardi et al. [10], who have computed the solutions of diffusion-wave equations of fractional order by using their Mellin–Barnes integral representation.

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Appendix. Some generalizations of the hypergeometric functions

The purpose of this Appendix is to provide a survey of some higher transcendental functions that have been proposed for generalizing the hypergeometric function. In particular, we shall consider the functions investigated by Pochhammer (1870) and Goursat (1883), that have interested Pincherle in his 1888 paper, and the G functions introduced by Meijer (1936–1946), since they are re-derived in our present analysis by extending the arguments by Pincherle. Our survey is essentially based on the classical handbook of the Bateman Project [5] and on the more recent treatise by Kiryakova [9].

Let us start by recalling the classical *hypergeometric equation*. If a homogeneous linear differential equation of the second order has at most three singular points we may assume that these are $0, 1, \infty$. If all these singular points are "regular", then the equation can be reduced to the form

$$z(1-z)\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}u}{\mathrm{d}z} - abu(z) = 0, \tag{A.1}$$

where a, b, c are arbitrary complex constants. This is the *hypergeometric equation*. Taking $c \neq 0, -1, -2, \ldots$, and defining the Pochhammer symbol

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad \text{i.e. } (\alpha)_0 = 1,\ (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1),\ n = 1, 2, \ldots$$

then the solution of Eqs. (A.1), regular at $z = 0$, known as *Gauss hypergeometric function*, turns out to be

$$u(z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n := F(a, b; c; z). \tag{A.2}$$

The above hypergeometric series can be generalized by introducing p parameters $a_1, \ldots, a_p$ (the numerator-parameters) and q parameters $b_1, \ldots, b_q$ (the denominator-parameters). The ensuing series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} := {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z), \tag{A.3}$$

or, in a more compact form,

$$\sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{p}(a_j)_n}{\Pi_{k=1}^{q}(b_j)_n} \frac{z^n}{n!} := {}_pF_q[(a_j)_1^p;(b_k)_1^q;z)] \tag{A.3$'$}$$

is known as the *generalized hypergeometric series*. In general (excepting certain integer values of the parameters for which the series fails to make sense or terminates[4]) the series ${}_pF_q$ converges for all finite z if $p \leqslant q$, converges for $|z|<1$ if $p=q+1$, and diverges for all $z \neq 0$ if $p > q+1$. The resulting *generalized hypergeometric function* $u(z) = {}_pF_q$ will satisfy a *generalized hypergeometric equation*. If we note that Eq. (A.1) satisfied by $u(z)=F(a,b;c;z)={}_2F_1(a,b;c;z)$ can be written in the equivalent form (see e.g. Rainville [25, Section 46, p. 75]):

$$\left[z\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b\right)-z\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+c-1\right)\right]u(z)=0, \tag{A.1$'$}$$

we arrive at the equation of order $n=q+1$ for $u(z) = {}_pF_q[(a_j)_1^p;(b_k)_1^q;z)]$:

$$\left[z\prod_{j=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a_j\right)-z\frac{\mathrm{d}}{\mathrm{d}z}\prod_{k=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+b_k-1\right)\right]u(z)=0. \tag{A.4}$$

The above equation containing the operator $z\,\mathrm{d}/\mathrm{d}z$ can be written in a more explicit form by using $D=\mathrm{d}/\mathrm{d}z$, see e.g. [5, Section 42, p. 184]. Distinguishing between the cases $p \leqslant q$ and $p=q+1$, we get the following general equations in $v=v(z)$:

$$z^q D^{q+1} v + \sum_{v=1}^{q} z^{v-1}(A_v z - B_v) D^v v + A_0 v = 0, \quad p \leqslant q, \tag{A.5}$$

$$z^q (1-z) D^{q+1} v + \sum_{v=1}^{q} z^{v-1}(A_v z - B_v) D^v v + A_0 v = 0, \quad p = q+1, \tag{A.6}$$

where A_0, A_v, B_v are constants. Eq. (A.5) has two singular points, $z=0,\infty$ of which $z=0$ is of regular type, whereas Eq. (A.6) has three singular points, $z=0,1,\infty$ of regular type, like Eq. (A.1). An equation of the same type as Eq. (A.6) was formerly introduced in 1883 by Goursat [7,8] in his essay on hypergeometric functions of higher order.

Another generalization of the Gauss hypergeometric equation was previously proposed in 1870 by Pochhammer [23]. He investigated the most general homogeneous linear differential equation of the order n $(n>2)$ of Fuchsian type, namely with only "regular" singular points in $\{a_1, a_2, \ldots, a_n, \infty\}$. The Pochhammer function thus satisfies a differential equation of the type

$$\phi_n(z)\frac{\mathrm{d}^n w}{\mathrm{d}z^n} + \cdots \phi_1(z)\frac{\mathrm{d}w}{\mathrm{d}z} + \phi_0 w(z) = 0, \tag{A.7}$$

[4] If at least one of the denominator parameters b_k $(k=1,\ldots,q)$ is zero or a negative integer, Eq. (A.3) has no meaning at all, since the denominator of the general term vanishes for a sufficiently large index. If some of the numerator parameters are zero or negative integers, then the series terminates and turns into a *hypergeometric polynomial*.

where the coefficients $\phi_v(z)$ $(v=0,1,\ldots,n)$ are polynomials of degree v, with $\phi_n(z)=(z-a_1)(z-a_2)\cdots(z-a_n)$.

The ${}_pF_q$ functions satisfying Eqs. (A.5) and (A.6) and the Pochhammer functions satisfying Eq. (A.7) are not the only generalizations of the Gauss hypergeometric function (A.2). In 1936, Meijer [12] introduced a new class of transcendental functions, the so-called G functions, which provide an interpretation of the symbol ${}_pF_q$ when $p>q+1$. Originally, the G function was defined in a manner resembling (A.2). Later [13], this definition was replaced by one in terms of Mellin–Barnes type integrals. The latter definition has the advantage that it allows a greater freedom in the relative values of p and q. Here, following [5], we shall complete Meijer's definition so as to include all values of p and q without placing any (non-trivial) restriction on m and n. One defines

$$G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]=G^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j)_1^p\\ (b_j)_1^q\end{matrix}\right]=\frac{1}{2\pi \mathrm{i}}\int_L \mathcal{G}^{m,n}_{p,q}(s)z^s\,\mathrm{d}s, \tag{A.8}$$

where L is a suitably chosen path, $z\neq 0$, $z^s := \exp[s(\ln|z|+\mathrm{i}\arg z)]$ with a single valued branch of $\arg z$, and the integrand is defined as follows:

$$\mathcal{G}^{m,n}_{p,q}(s)=\frac{\prod_{k=1}^m \Gamma(b_k-s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{k=m+1}^q \Gamma(1-b_k+s)\prod_{j=n+1}^p \Gamma(a_j-s)}. \tag{A.9}$$

In (A.9), an empty product is interpreted as 1, the integers m,n,p,q (known as *orders of the G function*) are such that $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$, and the parameters a_j and b_k are such that no pole of $\Gamma(b_k-s)$, $k=1,\ldots,m$, coincides with any pole of $\Gamma(1-a_j+s)$, $j=1,\ldots,n$. For the details of the integration path, which can be of three different types, we refer to [5] (see also [9] where an illustration of what these contours can be like is found).

One can establish that the Meijer G function $u(z)$ satisfies the linear ordinary differential equation of generalized hypergeometric type, see e.g. [9, p. 316, Eq. (A.19)],

$$\left[(-1)^{p-m-n}z\prod_{j=1}^p\left(z\frac{\mathrm{d}}{\mathrm{d}z}-a_j+1\right)-\prod_{k=1}^q\left(z\frac{\mathrm{d}}{\mathrm{d}z}-b_k\right)\right]u(z)=0. \tag{A.10}$$

For more details on the Meijer function and on the singular points of the above differential equation we refer to [9]. Here, we limit ourselves to show how the generalized hypergeometric function ${}_pF_q$ can be expressed in terms of a Meijer G function and thus in terms of Mellin–Barnes integral. We have

$${}_pF_q((a)_p;(b)_q;z)=\frac{\Pi_{k=1}^q\Gamma(b_k)}{\Pi_{j=1}^p\Gamma(a_j)}\,G^{1,p}_{p,q+1}\left[-z\,\middle|\,\begin{matrix}(1-a_j)_1^p\\ 0,(1-b_k)_1^q\end{matrix}\right], \tag{A.11}$$

$$G^{1,p}_{p,q+1}=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\infty}\frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}(-z)^s\,\mathrm{d}s, \tag{A.12}$$

$$a_j\neq 0,-1,-2,\ldots;\quad j=1,\ldots,p;\quad |\arg(1-z\mathrm{i})|<\pi. \tag{A.13}$$

Here, the path of integration is the imaginary axis (in the complex s-plane) which can be deformed, if necessary, in order to separate the poles of $\Gamma(a_j+s)$, $j=1,\ldots,p$ from those of $\Gamma(-s)$.

Though the G functions are quite general in nature, there still exist examples of special functions, like the Mittag–Leffler and the Wright functions, which do not form their particular cases. A more general class which includes those functions can be achieved by introducing the Fox H functions [6], whose representation in terms of the Mellin–Barnes integral is a straightforward generalization of that for the G functions. For this purpose, we need to add to the sets of the complex parameters a_j and b_k the new sets of positive numbers α_j and β_k with $j=1,\ldots,p$, $k=1,\ldots,q$, and modify in the integral of (A.8) the kernel $\mathscr{G}^{m,n}_{p,q}(s)$ into

$$\mathscr{H}^{m,n}_{p,q}(s)=\frac{\prod_{k=1}^{m}\Gamma(b_k-\beta_k s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+\beta_k s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}. \tag{A.14}$$

Then the Fox H function turns out to be defined as

$$H^{m,n}_{p,q}(z)=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{j=1,\ldots,p}\\(b_k,\beta_k)_{k=1,\ldots,q}\end{matrix}\right.\right]=\frac{1}{2\pi\mathrm{i}}\int_L \mathscr{H}^{m,n}_{p,q}(s)z^s\,\mathrm{d}s. \tag{A.15}$$

We do not pursue furthermore in our survey: we refer the interested reader to the treatises on Fox H functions by Mathai and Saxena [11], Srivastava et al. [26] and references therein.

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