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Lévy Stable Distributions in the Theory of Probability

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1 Introduction

Stable distributions are a fascinating and fruitful area of research in probability theory; furthermore, nowadays, they provide valuable models in physics, astronomy, economics, and communication theory, see *e.g.* ....

The general class of stable distributions was introduced and given this name by the French mathematician Paul Lévy in the early 1920's, see Lévy (1923,1924,1925).

Formerly, the topic attracted only moderate attention from the leading experts, though there were also enthusiasts, of whom the Russian mathematician Alexander Yakovlevich Khintchine should be mentioned first of all. The inspiration for Lévy was the desire to generalize the celebrated *Central Limit Theorem*, according to which any probability distribution with finite variance belongs to the domain of attraction of the Gaussian distribution. The concept of stable distributions took full shape in 1937 with the appearance of Lévy's monograph [63], soon followed by Khintchine's monograph [46][2].

[1] LaTeX file fm_stable-new.tex

[2] Nowadays F. Mainardi and S. Rogosin, with the aim of reevaluating the work of Khintchine on limit theorems of sums of independent variables (not necessarily identically distributed), are planning to translate into English (from the Russian, Italian, German and French) the relevant papers by Khintchine and also a number of related papers by De Finetti,

The theory and properties of stable distributions have been systematically presented by Gnedenko & Kolmogorov [26] and Feller [**?**]. These distribution are also discussed in some other classical books in probability theory including Lukacs (1960-1970), Feller (1966-1971), Breiman (1968-1992), Chung (1968-1974) and Laha & Rohatgi (1979). Also treatises on fractals devote particular attention to stable distributions in view of their properties of scale invariance, see *e.g.* Mandelbrot (1982) and Takayasu (1990). It is only recently that monographs devoted solely to stable distributions and related stochastic processes have been appeared, *i.e.* Zolotarev (1983-1986), Janicki & Weron (1994), and Samorodnitsky & Taqqu (1994), Uchaikin & Zolotarev (1999), Nolan (????). In these books tables and/or graphs related to stable distributions are also exhibited. Previous sets of tables and graphs have been provided by Mandelbrot & Zarnfaller (1959), Fama & Roll (1968), Bo'lshev & Al. (1968) and Holt & Crow (1973).

Stable distributions have three *exclusive* properties, which can be briefly summarized stating that they 1) are *invariant under addition*, 2) possess their own *domain of attraction*, and 3) admit a *canonical characteristic function.*

In the following sections let us illustrate the above properties which, providing necessary and sufficient conditions, can be assumed as equivalent definitions for a stable distribution. We recall the basic results without proof.

2 Invariance under addition

A random variable X *is said to have a stable distribution* $P(x) = \text{Prob}\,\{X \leq x\}$ *if for any* $n \geq 2$, *there is a positive number* c_n *and a real number* d_n *such that*

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} c_n\, X + d_n\,, \tag{2.1}$$

where $X_1, X_2, \ldots X_n$ *denote mutually independent random variables with common distribution* $P(x)$ *with* X. Here the notation $\stackrel{d}{=}$ denotes equality in distribution, *i.e.* means that the random variables on both sides have the same probability distribution.

When mutually independent random variables have a common distribution [shared with a given random variable X], we also refer to them as independent, identically distributed (i.i.d) random variables [independent copies of X]. In

Lévy, Kolmogorov, Gnedenko, Feller, that have been source of inspiration for Khintchine himself. The 1938 book by Khintchine has already been translated from the Russian by Rogosin general, the sum of i.i.d. random variables becomes a random variable with a distribution of different form. However, for independent random variables with a common *stable* distribution, the sum obeys to a distribution of the same type, which differs from the original one only for a scaling (c_n) and possibly for a shift (d_n). When in (A.1) the $d_n = 0$ the distribution is called *strictly stable.*

It is known, see [20], that the norming constants in (2.1) are of the form

$$c_n = n^{1/\alpha} \quad \text{with} \quad 0 < \alpha \le 2\,. \tag{2.2}$$

The parameter α is called the *characteristic exponent* or the *index of stability* of the stable distribution. We agree to use the notation $X \sim P_\alpha(x)$ to denote that the random variable X has a stable probability distribution with characteristic exponent α. We simply refer to $P_\alpha(x)$, $p_\alpha(x) := dP_\alpha(x)/dx$ (probability density functions = *pdf*) and X as *α-stable distribution, density, random variable*, respectively.

Definition (2.1) with theorem (2.2) can be stated in an alternative version that needs only two i.i.d. random variables. see also Lukacs (1960-1970). *A random variable X is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that*

$$A\,X_1 + B\,X_2 \stackrel{d}{=} C\,X + D\,, \tag{2.3}$$

where X_1 and X_2 are independent copies of X. Then there is a number $\alpha \in (0,2]$ such that the number C in (2.3) satisfies $C^\alpha = A^\alpha + B^\alpha$.

For a strictly stable distribution Eq. (2.3) holds with $D = 0$. This implies that all linear combinations of i.i.d. random variables obeying to a strictly stable distribution is a random variable with the same type of distribution.

A stable distribution is called *symmetric* if the random variable $-X$ has the same distribution. Of course, a *symmetric* stable distribution is necessarily *strictly stable.*

Noteworthy examples of stable distributions are provided by the Gaussian (or normal) law (with $\alpha = 2$) and by the Cauchy-Lorentz law ($\alpha = 1$). The corresponding *pdf*'s are known to be

$$p_G(x;\sigma,\mu) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\,, \quad x \in \mathbf{R}\,, \tag{2.4a}$$

where σ^2 denotes the variance and μ the mean, and

$$p_C(x;\gamma,\delta) := \frac{1}{\pi}\,\frac{\gamma}{(x-\delta)^2 + \gamma^2}\,, \quad x \in \mathbf{R}\,, \tag{2.5a}$$

where γ denotes the semi-interquartile range and δ the "shift". The corresponding (cumulative) distribution functions are

$$P_G(x;\sigma,) := \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{x}{\sqrt{2}\,\sigma}\right)\right], \quad x \in \mathbf{R}, \tag{2.4b}$$

and

$$P_C(x;\gamma,0) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{\gamma}\right), \quad x \in \mathbf{R}. \tag{2.5b}$$

3 Domain of attraction

Another (equivalent) definition states that stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d. random variables. A random variable X is said to have a *domain of attraction,i.e.* if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{\gamma_n\}$ and real numbers $\{\delta_n\}$, such that

$$\frac{Y_1 + Y_2 + \ldots Y_n}{\gamma_n} + \delta_n \stackrel{d}{\Rightarrow} X. \tag{3.1}$$

The notation $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution.

If the random variable X has a stable distribution and all Y_i are taken to be independent and distributed like X, then clearly (2.1) implies (3.1), and so trivially every random variable with a stable probability distribution has a domain of attraction. The converse is also true, namely that every random variable with a domain of attraction has a stable probability distribution, see [26]. Therefore we can alternatively state that *a random variable X is said to have a stable distribution if it has a domain of attraction.*

When X is Gaussian and the Y_i's are i.i.d. with finite variance, then (3.1) is the statement of the ordinary *Central Limit Theorem.* The domain of attraction of X is said *normal* when $\gamma_n = n^{1/\alpha}$; in general, $\gamma_n = n^{1/\alpha}\,h(n)$ where $h(x)$, $x > 0$, is a slow varying function at infinity[3] The function $h(x) = \ln x$, for example, is slowly varying at infinity: it enters in a general

[3]**Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, *slowly varying at zero* if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, *slowly varying at infinity* if $b(cy)/a(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $(\log y)^\gamma$ with $\gamma \in \mathbf{R}$ and $\exp(\log y/\log\log y)$.

theorem on the domain of attraction of the Gaussian law for distributions with infinite variance (in particular densities decaying as $|x|^{-3}$), as formerly shown independently by Khintchine, Lévy and Feller, see *e.g.* [46, 26, 20].

4 Canonical forms for the characteristic function

Another definition specifies the *canonical form* that the *characteristic function* (cf) of a stable distribution of index α must have. Let us recall that the cf is the Fourier transform of the *pdf*. If we denote by $p_\alpha(x;\cdot)$ the *pdf* for a generic stable distribution of index α (the $\cdot$ stands for additional parameters) the corresponding cf reads in our notation

$$\hat{p}_\alpha(\kappa;\cdot) := \langle \exp i\kappa X \rangle = \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, p_\alpha(x;\cdot)\, dx \div p_\alpha(x;\cdot)\,,$$

where $\div$ denotes the juxtaposition of a function with its Fourier transform.

We note that the cf of the most popular stable distributions, the Gaussian (2.4) and the Cauchy-Lorentz (2.5), turn out to be

$$\widehat{p}_G(\kappa;\sigma) = \mathrm{e}^{-(\sigma^2/2)\,|\kappa|^2}\,, \tag{4.1}$$

$$\widehat{p}_c(\kappa;\gamma) = \mathrm{e}^{-\gamma|\kappa|}\,. \tag{4.2}$$

Let first consider the (simplified) canonical form for *strictly stable distributions* adapting from that formerly introduced by Feller [19, 20] and Zolotarev [92, 90]. Using our notation this canonical forms reads

$$\hat{p}_\alpha(\kappa;\theta) := \exp\left\{-|\kappa|^\alpha\,\mathrm{e}^{\,i(\mathrm{sign}\ \kappa)\,\theta\,\pi/2}\right\}\,, \tag{4.3}$$

were θ is a real parameter whose domain is restricted to the following region (depending on α)

$$|\theta| \le \begin{cases} \alpha\,, & \text{if } 0<\alpha\le 1\,,\\ 2-\alpha\,, & \text{if } 1<\alpha\le 2\,. \end{cases} \tag{4.4}$$

We recognize that $p_\alpha(x,\theta) = p_\alpha(-x,-\theta)$, so the *symmetric* stable distributions are obtained if and only if $\theta = 0$; θ is called the asymmetry parameter, or simply (but improperly) *skewness*.

In the plane $\{\alpha\,,\,\theta\}$ the allowed region for the parameters α and θ $\{0 < \alpha \le 2\,,\, |\theta| \le\ \min(\alpha, 2-\alpha)\}$ turns out to be a diamond with vertices in the

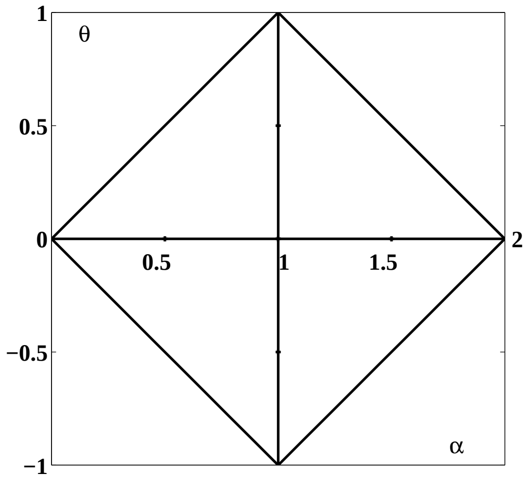

Figure 1: The Feller-Takayasu diamond

points $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$, that was formerly depicted in Takayasu's book, see Fig. 1. Honouring both Feller and Takayasu, we call the described region the *Feller-Takayasu diamond.*

We note that in his original and pioneering paper [19], Feller used a skewness parameter δ different from our θ ; in fact his characteristic function turns out to be

$$\hat{p}_\alpha^F(\kappa;\delta) := \exp\left\{-\left[|\kappa|\, \mathrm{e}^{-i\,(\mathrm{sign}\ \kappa)\,\delta}\right]^\alpha\right\}, \quad \text{so} \quad \delta = -\frac{\pi}{2}\,\frac{\theta}{\alpha}, \quad \theta = -\frac{2}{\pi}\,\alpha\delta\,. \tag{4.5}$$

In his two books [92, 90] Zolotarev used a notation which is confusing and misleading due to irritating misprints: as matter of fact one can recognizes that two different canonic forms are used for strictly stable distributions, namely

$$\hat{p}_\alpha^Z(\kappa;\delta_{1,2}) := \begin{cases} \exp\left\{-|\kappa|^\alpha\, \mathrm{e}^{-i(\mathrm{sign}\ \kappa)\,\delta_1\,\alpha\,\pi/2}\right\}, \\ \exp\left\{-|\kappa|^\alpha\, \mathrm{e}^{-i(\mathrm{sign}\ \kappa)\,\delta_2\,\pi/2}\right\}. \end{cases} \tag{4.6}$$

Our notation is in agreement with that adopted by Schneider (1986) [he, however, uses the letter β instead of θ] and Takayasu (1990) [see (5.23)-(5.24), p. 124], but all of them neglect the case $\alpha = 1$.

Feller has proved that, assuming $1/2 < \alpha < 1$ and $x > 0$,

$$\frac{1}{x^{\alpha+1}}\, p_{1/\alpha}(x^{-\alpha};\theta) = p_\alpha(x;\theta^*)\,, \quad \theta^* = \alpha(\theta+1) - 1\,. \tag{4.7}$$

A quick check shows that θ^* falls within the prescribed range, $|\theta^*| \le \alpha$, provided that $|\theta| \le 2 - 1/\alpha$.

Stable distributions with extremal values of the skewness parameter are called *extremal.* One can prove that all the extremal stable distributions with $0 < \alpha < 1$ are one-sided, the support being $\mathbf{R}_0^+$ if $\theta = -\alpha$, and $\mathbf{R}_0^-$ if $\theta = +\alpha$.

For $0 < \alpha < 2$ the stable distributions exhibit heavy tails in such a way that their absolute moment of order ν is finite only if $\nu < \alpha$. In fact one can show that for non-Gaussian, not extremal, stable distributions the asymptotic decay of the tails is

$$p_\alpha(x;\theta) = O\left(|x|^{-(\alpha+1)}\right), \quad x \to \pm\infty. \tag{4.8}$$

For the extremal distributions this is valid only for one tail, the other being of exponential order. For $0 < \alpha < 1$ we have one-sided distributions which exhibit an exponential left tail (as $x \to 0^+$) if $\theta = -\alpha$, or an exponential right tail (as $x \to 0^-$) if $\theta = +\alpha$. For $1 < \alpha < 2$ the extremal distributions are two-sided and exhibit an exponential left tail (as $x \to -\infty$) if $\theta = +(2-\alpha)$, or an exponential right tail (as $x \to +\infty$) if $\theta = -(2-\alpha)$.

Consequently, the Gaussian distribution is the unique stable distribution with finite variance. Furthermore, when $\alpha \le 1$, the first absolute moment $\langle |X| \rangle$ is infinite as well, so we need to use the median to characterize the expected value.

However, there is a fundamental property shared by all the stable distributions that we like to point out: for any α the corresponding stable *pdf* is *unimodal* and indeed *bell-shaped*, *i.e.* its n-th derivative has exactly n zeros, see Gawronski (1984).

From Lévy's times it is usual to adopt a more *general canonical form* for stable distributions that takes into account of a *scale parameter* $\gamma > 0$, of a *shift parameter* $\delta \in \mathbf{R}$ in addition to a *skewness parameter* $\beta \in \mathbf{R}$ restricted to be $|\beta| \le 1$. For this class of stable distributions, partly following the notation of Samorodnitsky & Taqqu (1994) and denoting by Y the random variable, we write $Y \sim Q_\alpha(y;\beta,\gamma,\delta)$ with characteristic function

$$\hat{q}_\alpha(\kappa;\beta,\gamma,\delta) = \exp\left\{i\delta\kappa - \gamma^\alpha\,|\kappa|^\alpha\,\left[1 + i\,(\mathrm{sign}\,\kappa)\,\beta\,\omega(|\kappa|,\alpha)\right]\right\}, \tag{4.9}$$

where

$$\omega(|\kappa|,\alpha) = \begin{cases} \tan(\alpha\,\pi/2), & \text{if} \quad \alpha \ne 1, \\ -(2/\pi)\ln|\kappa|, & \text{if} \quad \alpha = 1. \end{cases} \tag{4.10}$$

Consequently *a random variable* Y *is said to have a stable distribution if there are four real parameters* $\alpha, \beta, \gamma, \delta$ *with* $0 < \alpha \le 2$, $-1 \le \beta \le +1$, $\gamma > 0$, *such* *that its characteristic function has the canonical form (4.9)-(4.11).* We note that the parameter β appears with different signs for $\alpha \neq 1$ and $\alpha = 1$. This minor point has been the source of great confusion in the literature, see Hall (1980) for a discussion. The presence of the logarithm for $\alpha = 1$ is the source of many difficulties, so this case has often to be treated separately.

An interesting problem is related to the inclusion of the canonical form (4.3)-(4.4), which is valid only for strictly stable distributions in the more general canonical form (4.9)-(4.10) We first note that the two parameters γ and δ in (4.9), being related to a scale transformation and a translation respectively, are not so essential since they do not change the shape of the distribution. If we take $\gamma = 1$ and $\delta = 0$, we obtain the so-called *standardized* form of the stable distribution and the corresponding random variable $Y \sim P_\alpha(y;\beta,1,0)$ is referred to as the α-stable *standardized* random variable. On the other hand, keeping $\delta = 0$, we can choose the scale parameter γ in such a way to get the simplified canonical form for strictly stable distributions. As a matter of fact the relation between the two classes X and Y for stable random variables can be explored if we compare the corresponding canonical forms not only keeping $\delta = 0$ but also assuming $\alpha \neq 1$, as shown below.

We easily recognize

$$\gamma^\alpha = \cos\left(\theta\,\frac{\pi}{2}\right)\,, \quad \tan\left(\theta\,\frac{\pi}{2}\right) = \beta \tan\left(\alpha\,\frac{\pi}{2}\right)\,, \tag{4.11}$$

so the case $\alpha = 1$ must be excluded. Thus, the Feller-Takayasu canonical form for strictly stable distributions with index $\alpha \neq 1$ and skewness θ, can be obtained from the Lévy canonical form if we (implicitly) select the shift parameter $\delta = 0$, and the scale parameter γ and the skewness parameter β related to α and θ according to (4.11). We note that necessarily $0 < \gamma \leq 1$. We also note that for $\alpha = 1$ we have identity between the two canonical forms in the limiting case $\theta = \beta = 0$ corresponding to the (symmetric) Cauchy-Lorentz *pdf*. By the way for $\alpha = 1$ with $\delta = 0$ and $\beta \neq 0$ the general Lévy canonical form yields *not strictly* stable distributions.

Specifically, the random variable $X \sim P_\alpha(x;\theta)$ turns out to be related to the *standardized* random variable $Y \sim Q_\alpha(y;\beta,1,0)$ by the relations:

$$X = Y/\gamma\,, \qquad q_\alpha(y;\beta,1,0) = \gamma\, p_\alpha(x = \gamma y;\theta)\,, \tag{4.12}$$

with

$$\gamma = [\cos(\theta\pi/2)]^{1/\alpha}\,, \tag{4.13}$$

and

$$\theta = (2/\pi)\arctan\left[\beta\tan(\alpha\pi/2)\right]\,, \quad \beta = \frac{\tan(\theta\pi/2)}{\tan(\alpha\pi/2)}\,. \tag{4.14}$$

We note that for the *symmetric* stable distributions we get the identity between the *standardized* and the *Lévy* canonical forms, since in (4.14) $\beta = \theta = 0$ implies in (4.13) $\gamma = 1$. A particular but noteworthy case is provided by $p_2(x;0) = q_2(y;0,1,0)$ corresponding to the Gaussian distribution with variance $\sigma^2 = 2$. The identity is valid also in the limit for $\alpha = 1$, for which $p_1(x;0) = q_1(y;0,1,0)$ corresponding to the Cauchy-Lorentz distribution with semi-interquartile $\gamma = 1$.

The *extremal* stable distributions corresponding to $\beta = \pm 1$ and $\alpha \neq 1$ are obtained in the Feller-Takayasu representation for $\theta = \pm\alpha$ if $0 < \alpha < 1$, and for $\theta = \mp(2-\alpha)$ if $1 < \alpha < 2$. For these cases, the scaling parameter turns out to be $\gamma = [\cos(|\alpha|\,\pi/2)]^{1/\alpha}$.

It may be an instructive exercise to carry out the inversion of the Fourier transform when $\alpha = 1/2$ and $\theta = -1/2$. In this case we obtain the analytical expression for the corresponding extremal stable *pdf*, known as the (one-sided) *Lévy-Smirnov* density,

$$p_{1/2}(x;-1/2) = \frac{1}{2\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/(4x)}\,, \quad x \geq 0\,. \tag{4.15}$$

The corresponding *standardized* form for this distribution can be easily obtained from (4.15) using (4.12)-(4.14) with $\alpha = 1/2$ and $\theta = -1/2$. We get $\gamma = [\cos(-\pi/4)]^2 = 1/2$, $\beta = -1$, so

$$q_{1/2}(y;-1,1,0) = \frac{1}{2}\, p_{1/2}(y/2;-1/2) = \frac{1}{\sqrt{2\pi}}\, y^{-3/2}\, \mathrm{e}^{-1/(2y)}\,,\ y \geq 0\,, \tag{4.16}$$

in agreement with Holt & Crow (1973) [§2.13, p. 147].

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