Appendix B

The Bessel Functions

As Rainville pointed out in his classic booklet [Rainville (1960)], no other special functions have received such detailed treatment in readily available treatises as the Bessel functions. Consequently, we here present only a brief introduction to the subject including the related Laplace transform pairs used in this book.

B.1 The standard Bessel functions

The Bessel functions of the first and second kind: J_ν,Y_ν. The Bessel functions of the first kind $J_\nu(z)$ are defined from their power series representation:

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu} , \qquad (B.1)$$

where z is a complex variable and ν is a parameter which can take arbitrary real or complex values. When ν is integer it turns out as an entire function; in this case

$$J_{-n}(z) = (-1)^n \, J_n(z) \, , \quad n = 1, 2, \ldots \qquad (B.2)$$

In fact

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n} ,$$

$$J_{-n}(z) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!(k-n)!} \left(\frac{z}{2}\right)^{2k-n} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!s!} \left(\frac{z}{2}\right)^{2s+n} .$$

When ν is not integer the Bessel functions exhibit a branch point at $z = 0$ because of the factor $(z/2)^\nu$, so z is intended with $|\arg(z)| < \pi$ that is in the complex plane cut along the negative real semi-axis. Following a suggestion by Tricomi, see [Gatteschi (1973)], we can extract from the series in (B.1) that singular factor and set:

$$J_\nu^T(z) := (z/2)^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k} . \qquad (B.3)$$

The entire function $J_\nu^T(z)$ was referred to by Tricomi as the *uniform Bessel function*. In some textbooks on special functions, see e.g. [Kiryakova (1994)], p. 336, the related entire function

$$J_\nu^C(z) := z^{-\nu/2} J_\nu(2z^{1/2}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(k+\nu+1)} \qquad (B.4)$$

is introduced and named the *Bessel-Clifford function*.

Since for fixed z in the cut plane the terms of the series (B.1) are analytic function of the variable ν, the fact that the series is uniformly convergent implies that the Bessel function of the first kind $J_\nu(z)$ is an entire function of order ν.

The Bessel functions are usually introduced in the framework of the Fucks–Frobenius theory of the second order differential equations of the form

$$\frac{d^2}{dz^2} u(z) + p(z)\,\frac{d}{dz} u(z) + q(z)\,u(z) = 0\,, \qquad (B.5)$$

where $p(z)$ and $q(z)$ are assigned analytic functions. If we chose in (B.5)

$$p(z) = \frac{1}{z}\,,\ q(z) = 1 - \frac{\nu^2}{z^2}\,, \qquad (B.6)$$

and solve by power series, we would just obtain the series in (B.1). As a consequence, we say that the Bessel function of the first kind satisfies the equation

$$u''(z) + \frac{1}{z}\,u'(z) + \left(1 - \frac{\nu^2}{z^2}\right)\,u(z) = 0\,, \qquad (B.7)$$

where, for shortness we have used the apices to denote differentiation with respect to z. It is customary to refer to Eq. (B.7) as the *Bessel differential equation*.

When ν is not integer the general integral of the Bessel equation is

$$u(z) = \gamma_1 \, J_\nu(z) + \gamma_2 \, J_{-\nu}(z) \, , \quad \gamma_1, \gamma_2 \in \mathbb{C} \, , \qquad (B.8)$$

since $J_{-\nu}(z)$ and $J_\nu(z)$ are in this case linearly independent with Wronskian

$$W\{J_\nu(z), J_{-\nu}(z)\} = -\frac{2}{\pi z} \, \sin(\pi\nu) \, . \qquad (B.9)$$

We have used the notation $W\{f(z), g(z)\} := f(z)\, g'(z) - f'(z)\, g(z)$.

In order to get a solution of Eq. (B.7) that is linearly independent from J_ν also when $\nu = n$ $(n = 0, \pm 1, \pm 2 \dots)$ we introduce the Bessel function of the second kind

$$Y_\nu(z) := \frac{J_{-\nu}(z) \, \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \, . \qquad (B.10)$$

For integer ν the R.H.S of (B.10) becomes indeterminate so in this case we define $Y_n(z)$ as the limit

$$Y_n(z) := \lim_{\nu \to n} Y_\nu(z) = \frac{1}{\pi} \left[\left. \frac{\partial J_\nu(z)}{\partial \nu} \right|_{\nu=n} - (-1)^n \left. \frac{\partial J_{-\nu}(z)}{\partial \nu} \right|_{\nu=n} \right] . \qquad (B.11)$$

We also note that (B.11) implies

$$Y_{-n}(z) = (-1)^n \, Y_n(z) \, . \qquad (B.12)$$

Then, when ν is an arbitrary real number, the general integral of Eq. (B.7) is

$$u(z) = \gamma_1 \, J_\nu(z) + \gamma_2 \, Y_\nu(z) \, , \quad \gamma_1, \gamma_2 \in \mathbb{C} \, , \qquad (B.13)$$

and the corresponding Wronskian turns out to be

$$W\{J_\nu(z), Y_\nu(z)\} = \frac{2}{\pi z} \, . \qquad (B.14)$$

The Bessel functions of the third kind: $H_\nu^{(1)}, H_\nu^{(2)}$. In addition to the Bessel functions of the first and second kind it is customary to consider the Bessel function of the third kind, or Hankel functions, defined as

$$H_\nu^{(1)}(z) := J_\nu(z) + iY_\nu(z) \, , \quad H_\nu^{(2)}(z) := J_\nu(z) - iY_\nu(z) \, . \qquad (B.15)$$

These functions turn to be linearly independent with Wronskian

$$W\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\} = -\frac{4i}{\pi z} \, . \qquad (B.16)$$

Using (B.10) to eliminate $Y_n(z)$ from (B.15), we obtain

$$\begin{cases} H_\nu^{(1)}(z) := \dfrac{J_{-\nu}(z) - \mathrm{e}^{-i\nu\pi}\, J_\nu(z)}{i \sin(\nu\pi)}\,, \\ H_\nu^{(2)}(z) := \dfrac{\mathrm{e}^{+i\nu\pi}\, J_\nu(z) - J_{-\nu}(z)}{i \sin(\nu\pi)}\,, \end{cases} \qquad (B.17)$$

which imply the important formulas

$$H_{-\nu}^{(1)}(z) = \mathrm{e}^{+i\nu\pi}\, H_\nu^{(1)}(z)\,, \quad H_{-\nu}^{(2)}(z) = \mathrm{e}^{-i\nu\pi}\, H_\nu^{(2)}(z)\,. \qquad (B.18)$$

The recurrence relations for the Bessel functions. The functions $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ satisfy simple *recurrence relations.* Denoting any one of them by $\mathcal{C}_\nu(z)$ we have:

$$\begin{cases} \mathcal{C}_\nu(z) = \dfrac{z}{2\nu}\,[\mathcal{C}_{\nu-1}(z) + \mathcal{C}_{\nu+1}(z)]\,, \\ \\ \mathcal{C}'_\nu(z) = \dfrac{1}{2}\,[\mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z)]\,. \end{cases} \qquad (B.19)$$

In particular we note

$$J'_0(z) = -J_1(z)\,, \quad Y'_0(z) = -Y_1(z)\,.$$

We note that $\mathcal{C}_\nu$ stands for *cylinder function*, as it is usual to call the different kinds of Bessel functions. The origin of the term *cylinde*r is due to the fact that these functions are encountered in studying the boundary–value problems of potential theory for cylindrical coordinates.

A more general differential equation for the Bessel functions. The differential equation (B.7) can be generalized by introducing three additional complex parameters λ, p, q in such a way

$$z^2 w''(z) + (1-2p)\, z w'(z) + \left(\lambda^2 q^2 z^{2q} + p^2 - \nu^2 q^2\right)\, w(z) = 0\,. \quad (B.20)$$

A particular integral of this equation is provided by

$$w(z) = z^p\, \mathcal{C}_\nu\left(\lambda\, z^q\right)\,. \qquad (B.21)$$

We see that for $\lambda = 1$, $p = 0$, $q = 1$ we recover Eq. (B.7).

The asymptotic representations for the Bessel functions. The asymptotic representations of the standard Bessel functions for $z \to 0$ and $z \to \infty$ are provided by the first term of the convergent series expansion around $z = 0$ and by the first term of the asymptotic series expansion for $z \to \infty$, respectively.

For $z \to 0$ (with $|\arg(z)| < \pi$ if ν is not integer) we have:

$$\begin{cases} J_{\pm n}(z) \sim (\pm 1)^n \dfrac{(z/2)^n}{n!}\,, \quad n = 0, 1, \dots, \\ J_\nu(z) \sim \dfrac{(z/2)^\nu}{\Gamma(\nu+1)}\,, \quad \nu \neq \pm 1, \pm 2 \dots . \end{cases} \tag{B.22}$$

$$\begin{cases} Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim \dfrac{2}{\pi} \log(z)\,, \\ \\ Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -\dfrac{1}{\pi}\,\Gamma(\nu)(z/2)^{-\nu},\ \nu > 0. \end{cases} \tag{B.23}$$

For $z \to \infty$ with $|\arg(z)| < \pi$ and for any ν we have:

$$\begin{cases} J_\nu(z) \sim \sqrt{\dfrac{2}{\pi z}} \cos\left(z - \nu\dfrac{\pi}{2} - \dfrac{\pi}{4}\right), \\ Y_\nu(z) \sim \sqrt{\dfrac{2}{\pi z}} \sin\left(z - \nu\dfrac{\pi}{2} - \dfrac{\pi}{4}\right), \\ H_\nu^{(1)}(z) \sim \sqrt{\dfrac{2}{\pi z}}\, \mathrm{e}^{+i\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}, \\ H_\nu^{(2)}(z) \sim \sqrt{\dfrac{2}{\pi z}}\, \mathrm{e}^{-i\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}. \end{cases} \tag{B.24}$$

The generating function of the Bessel functions of integer order. The Bessel functions of the first kind $J_n(z)$ are simply related to the coefficients of the Laurent expansion of the function

$$w(z,t) = \mathrm{e}^{z(t-1/t)/2} = \sum_{n=-\infty}^{+\infty} c_n(z) t^n\,, \quad 0 < |t| < \infty\,. \tag{B.25}$$

To this aim we multiply the power series of $\mathrm{e}^{zt/2}$, $\mathrm{e}^{-z/(2t)}$, and, after some manipulation, we get

$$w(z,t) = \mathrm{e}^{z(t-1/t)/2} = \sum_{n=-\infty}^{+\infty} J_n(z) t^n\,, \quad 0 < |t| < \infty\,. \tag{B.26}$$

The function $w(z,t)$ is called the *generating function* of the Bessel functions of integer order, and formula (B.26) plays an important role in the theory of these functions.

Plots of the Bessel functions of integer order. Plots of the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ for integer orders $\nu = 0, 1, 2, 3, 4$ are shown in Fig. B.1 and in Fig. B.2, respectively.

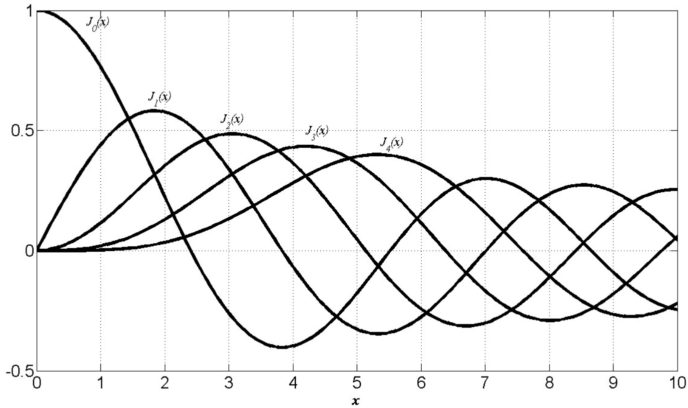

Fig. B.1 Plots of $J_\nu(x)$ with $\nu = 0, 1, 2, 3, 4$ for $0 \leq x \leq 10$.

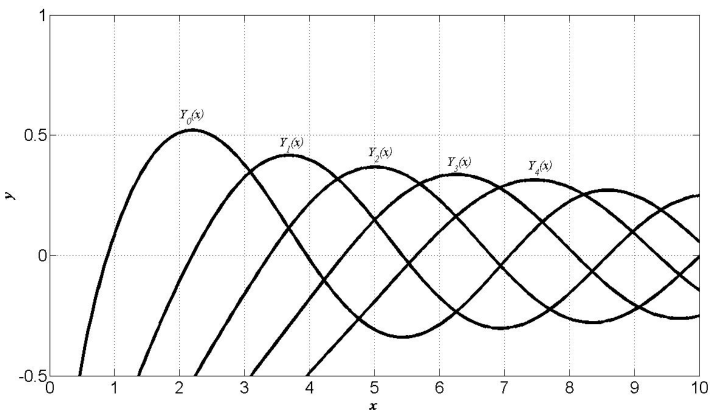

Fig. B.2 Plots of $Y_\nu(x)$ with $\nu = 0, 1, 2, 3, 4$ for $0 \leq x \leq 10$.

The Bessel functions of semi-integer order. We now consider the special cases when the order is a a semi-integer number $\nu = n + 1/2$ $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$. In these cases the standard Bessel function can be expressed in terms of elementary functions.

In particular we have

$$J_{+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z\,, \quad J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z\,. \quad (B.27)$$

The fact that any Bessel function of the first kind of half-integer order can be expressed in terms of elementary functions now follows from the first recurrence relation in (B.19), i.e.

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z}\, J_\nu(z)\,,$$

whose repeated applications gives

$$\begin{cases} J_{+3/2}(z) = \left(\dfrac{2}{\pi z}\right)^{1/2} \left[\dfrac{\sin z}{z} - \cos z\right], \\ J_{-3/2}(z) = -\left(\dfrac{2}{\pi z}\right)^{1/2} \left[\sin z - \dfrac{\cos z}{z}\right], \end{cases} \quad (B.28)$$

and so on.

To derive the corresponding formulas for Bessel functions of the second and third kind we start from the expressions (B.10) and (B.15) of these functions in terms of the Bessel functions of the first kind, and use (B.25). For example, we have:

$$Y_{1/2}(z) = -J_{-1/2}(z) = -\left(\frac{2}{\pi z}\right)^{1/2} \cos z\,, \quad (B.29)$$

$$H^{(1)}_{1/2}(z) = -i\left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{+iz}\,,\ H^{(2)}_{1/2}(z) = +i\left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{-iz}\,. \quad (B.30)$$

It has been shown by Liouville that the case of half-integer order is the only case where the cylinder functions reduce to elementary functions.

It is worth noting that when $\nu = \pm 1/2$ the asymptotic representations (B.24) for $z \to \infty$ for all types of Bessel functions reduce to the exact expressions of the corresponding functions provided above. This could be verified by using the saddle-point method for the complex integral representation of the Bessel functions, that we will present in Subsection B.3.

B.2 The modified Bessel functions

The modified Bessel functions of the first and second kind: I_ν, K_ν. The modified Bessel functions of the first kind $J_\nu(z)$ with $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ are defined by the power series

$$I_\nu(z) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu} . \qquad (B.31)$$

We also define the modified Bessel functions of the second kind $K_\nu(z)$:

$$K_\nu(z) := \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} . \qquad (B.32)$$

For integer ν the R.H.S of (B.32) becomes indeterminate so in this case we define $Y_n(z)$ as the limit

$$K_n(z) := \lim_{\nu \to n} K_\nu(z) . \qquad (B.33)$$

Repeating the consideration of Section B.1, we find that $I_\nu(z)$ and $K_\nu(z)$ are analytic functions of z in the cut plane and entire function of the order ν. We recall that $K_\nu(z)$ is sometimes referred to as *Macdonald's function.* We note from the definitions (B.31) and (B.32) the useful formulas

$$I_{-n}(z) = I_n(z) , \quad n = 0, \pm 1, \pm 2, \ldots \qquad (B.33)$$

$$K_{-\nu}(z) = K_\nu(z) , \quad \forall \nu . \qquad (B.34)$$

The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are simply related to the standard Bessel function of argument $z\exp(\pm i\pi/2)$. If

$$-\pi < \arg(z) < \pi/2 , \quad \text{i.e.,} \quad -\pi/2 < \arg(z\,\mathrm{e}^{i\pi/2}) < \pi/2 ,$$

then (B.1) implies

$$I_\nu(z) = \mathrm{e}^{-i\nu\pi/2}\, J_\nu(z\,\mathrm{e}^{i\pi/2}) . \qquad (B.34)$$

Similarly, according to (B.17), for the same value of z we have

$$K_\nu(z) = \frac{i\pi}{2}\, \mathrm{e}^{i\nu\pi/2}\, H^1_\nu(z\,\mathrm{e}^{i\pi/2}) . \qquad (B.35)$$

On the other hand, if

$$-\pi/2 < \arg(z) < \pi , \quad \text{i.e.,} \quad -\pi < \arg(z\,\mathrm{e}^{-i\pi/2}) < \pi/2 ,$$

then it is easily verified that

$$I_\nu(z) = \mathrm{e}^{+i\nu\pi/2}\, J_\nu(z\,\mathrm{e}^{-i\pi/2}) , \qquad (B.36)$$

and

$$K_\nu(z) = -\frac{i\pi}{2}\, \mathrm{e}^{-i\nu\pi/2}\, H^2_\nu(z\,\mathrm{e}^{-i\pi/2}) . \qquad (B.37)$$

The differential equation for the modified Bessel functions. It is an immediate consequence of their definitions that $I_\nu(z)$ and $K_\nu(z)$ are linearly independent solutions of the differential equation

$$v''(z) + \frac{1}{z}\, v'(z) - \left(1 + \frac{\nu^2}{z^2}\right) v(z) = 0\,, \qquad (B.38)$$

which differs from the standard Bessel equation (B.7) only by the sign of one term, and reduces to Eq. (B.7) if in Eq. (B.38) we make the substitution $z = \pm it$. Like the standard Bessel equation, Eq. (B.38) is often encountered in Mathematical Physics and it is referred to as the *modified Bessel differential equation.* Its general solution, for arbitrary ν can be written in the form

$$v(z) = \gamma_1\, I_\nu(z) + \gamma_2\, K_\nu(z)\,, \quad \gamma_1, \gamma_2 \in \mathbb{C}\,. \qquad (B.39)$$

For the modified Bessel functions the corresponding Wronskian turns out to be

$$W\{I_\nu(z), K_\nu(z)\} = -\frac{1}{z}\,. \qquad (B.40)$$

The recurrence relations for the modified Bessel functions. Like the cylinder functions, the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ satisfy simple *recurrence relations.* However, at variance with the cylinder functions, we have to keep distinct the corresponding recurrence relations:

$$\begin{cases} I_\nu(z) = \dfrac{z}{2\nu}\,[I_{\nu-1}(z) - I_{\nu+1}(z)]\,, \\[2ex] I'_\nu(z) = \dfrac{1}{2}\,[I_{\nu-1}(z) + I_{\nu+1}(z)]\,, \end{cases} \qquad (B.41)$$

and

$$\begin{cases} K_\nu(z) = -\dfrac{z}{2\nu}\,[K_{\nu-1}(z) - K_{\nu+1}(z)]\,, \\[2ex] K'_\nu(z) = -\dfrac{1}{2}\,[K_{\nu-1}(z) + K_{\nu+1}(z)]\,. \end{cases} \qquad (B.42)$$

Recurrence relations (B.41) and (B.42) can be written in a unified form if, following [Abramowitz and Stegun (1965)], we set

$$\mathscr{Z}_\nu(z) := \left\{I_\nu(z), \mathrm{e}^{i\nu\pi}\, K_\nu(z)\right\}\,. \qquad (B.43)$$

In fact we get

$$\begin{cases} \mathcal{Z}_\nu(z) = \dfrac{z}{2\nu}\left[\mathcal{Z}_{\nu-1}(z) - \mathcal{Z}_{\nu+1}(z)\right], \\ \\ \mathcal{Z}'_\nu(z) = \dfrac{1}{2}\left[\mathcal{Z}_{\nu-1}(z) + \mathcal{Z}_{\nu+1}(z)\right], \end{cases} \qquad (B.44)$$

which preserves the form of (B.41).

A more general differential equation for the modified Bessel functions. As for the standard Bessel functions we have provided the reader with a more general differential equation solved by related functions, see (B.20) and (B.21), so here we do it also for the modified Bessel functions. For this purpose it is sufficient to replace there λ^2 with $-\lambda^2$. Then, introducing three additional complex parameters λ, p, q in such a way that

$$z^2 w''(z) + (1-2p) z w'(z) + \left(-\lambda^2 q^2 z^{2q} + p^2 - \nu^2 q^2\right) w(z) = 0\,, \quad (B.45)$$

we get the required differential equation whose a particular integral is provided by

$$w(z) = z^p\, \mathcal{Z}_\nu\left(\lambda\, z^q\right)\,. \qquad (B.46)$$

Note that for $\lambda = 1$, $p = 0$, $q = 1$ in (B.45) we recover Eq. (B.38). Of course the constant $\mathrm{e}^{i\nu\pi}$ multiplying the function $K_\nu(z)$ is not relevant for Eqs. (B.45)-(B.46), but it is essential to preserve the same form for the recurrence relations satisfied by the two functions denoted by $\mathcal{Z}_\nu(z)$, as shown in Eqs. (B.44).

The asymptotic representations for the modified Bessel functions. For the modified Bessel functions we have the following asymptotic representations as $z \to 0$ and as $z \to \infty$.

For $z \to 0$ (with $|\arg(z)| < \pi$ if ν is not integer) we have:

$$\begin{cases} I_{\pm n}(z) \sim \dfrac{(z/2)^n}{n!}\,, \quad n = 0, 1, \ldots, \\ \\ I_\nu(z) \sim \dfrac{(z/2)^\nu}{\Gamma(\nu+1)}\,, \quad \nu \neq \pm 1, \pm 2 \ldots. \end{cases} \qquad (B.47)$$

and

$$\begin{cases} K_0(z) \sim \log{(2/z)}\,, \\ K_\nu(z) \sim \dfrac{1}{2}\,\Gamma(\nu)\,(z/2)^{-\nu}\,,\ \nu > 0\,. \end{cases} \qquad (B.48)$$

For $z \to \infty$ with $|\arg(z)| < \pi/2$ and for any ν we have:

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi}}\, z^{-1/2}\, \mathrm{e}^{\,z}\,, \qquad (B.49)$$

$$K_\nu(z) \sim \frac{1}{\sqrt{2\pi}}\, z^{-1/2}\, \mathrm{e}^{\,-z}\,. \qquad (B.50)$$

The generating function of the modified Bessel functions of integer order. For the modified Bessel functions of the first kind $I_n(z)$ of integer order we can establish a generating function following a procedure similar to that adopted for $J_n(z)$, see Eqs. (B.25)-(B.26). In fact, by considering the Laurent expansion of the function $\omega(z,t) = \mathrm{e}^{z(t+1/t)/2}$ obtained by multiplying the power series of $\mathrm{e}^{zt/2}$, $\mathrm{e}^{z/(2t)}$, we get after some manipulation

$$\omega(z,t) = \mathrm{e}^{z(t+1/t)/2} = \sum_{n=-\infty}^{+\infty} I_n(z)\, t^n\,, \quad 0 < |t| < \infty\,. \qquad (B.51)$$

Plots of the modified Bessel functions of integer order. Plots of the Bessel functions $I_\nu(x)$ and $K_\nu(x)$ for integer orders $\nu = 0, 1, 2$ are shown in Fig. B.3.

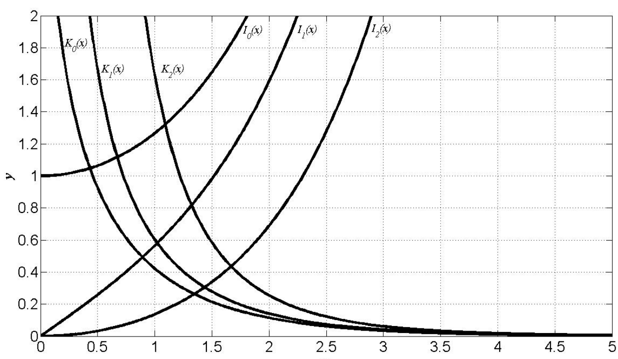

Fig. B.3 Plots of $I_\nu(x)$, $K_\nu(x)$ with $\nu = 0, 1, 2$ for $0 \le x \le 5$.

Since the modified Bessel functions exhibit an exponential behaviour for $x \to \infty$, see (B.49)-(B.50), we show the plots of $\mathrm{e}^{-x}\, I_\nu(x)$ and $\mathrm{e}^{x}\, K_\nu$ with $\nu = 0, 1, 2$ for $0 \le x \le 5$ in Fig B.4.

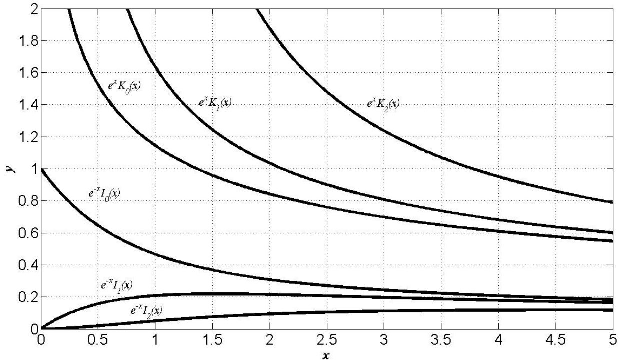

Fig. B.4 Plots of $\mathrm{e}^{-x} I_\nu(x)$, $\mathrm{e}^{x}\, K_\nu$ with $\nu = 0, 1, 2$ for $0 \le x \le 5$.

The modified Bessel functions of semi-integer order. Like the cylinder functions the modified Bessel functions of semi-integer order can be expressed in terms of elementary functions. Starting with the case $\nu = 1/2$ it is easy to recognize

$$I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sinh z\,, \quad I_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cosh z\,, \quad (B.52)$$

and

$$K_{1/2}(z) = K_{-1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}\,. \quad (B.53)$$

For general index $\nu = n + 1/2$ the corresponding formulas are obtained from (B.48) and the recurrence relations (B.41) and (B.42).

B.3 Integral representations and Laplace transforms

Integral representations. The basic integral representation of the standard Bessel function $J_\nu(z)$ with $\mathcal{R}e\, z > 0$ is provided through the Hankel contour around the negative real axis, denoted by Ha_- and illustrated in Fig. A.3 left. We have, see e.g. [Davies (2002)],

$$J_\nu(z) = \frac{1}{2\pi i} \int_{Ha_-} p^{-\nu - 1} \exp\left[\frac{z}{2}\left(p - \frac{1}{p}\right)\right] dp\,, \; \mathcal{R}e\, z > 0\,, \quad (B.54)$$

where the restriction $\mathcal{R}e\, z > 0$ is necessary to make the integral converge. Then, we can split the integral in two contributions:
(i) from the circular path where $p = \exp(i\theta)$ $(-\pi < \theta < \pi)$;
(ii) from the straight paths where $p = \exp(s \pm i\pi)$ $(0 < s < \infty)$.

We have:

$$(i)\ \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{e}^{(-i\nu\theta + iz\sin\theta)}\, d\theta = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z\sin\theta)\, d\theta\,; \qquad (B.55)$$

$$(ii)\ \frac{1}{2\pi i} \int_\infty^0 \mathrm{e}^{\nu s + i\pi\nu - z(\mathrm{e}^s - \mathrm{e}^{-s})/2}\, ds + \frac{1}{2\pi i} \int_\infty^0 \mathrm{e}^{\nu s - i\pi\nu - z(\mathrm{e}^s - \mathrm{e}^{-s})/2}\, ds$$

$$= -\frac{\sin(\nu\pi)}{\pi} \int_0^\infty \exp(-z\sinh s - \nu s)\, ds\,. \qquad (B.56)$$

Thus, the final integral representation is

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z\sin\theta)\, d\theta$$
$$- \frac{\sin(\nu\pi)}{\pi} \int_0^\infty \exp(-z\sinh s - \nu s)\, ds\,, \; \mathcal{R}e\, z > 0\,. \qquad (B.57)$$

For integer ν the second integral vanishes. The first integral is known as Bessel's integral.

It is a simple matter to perform an analytic continuation of (B.54) to all the domain of analyticity of $J_\nu(z)$. If we temporarily restrict z to be real and positive, then the change of variables $u = pz/2$ yields

$$J_\nu(z) = \frac{(z/2)^\nu}{2\pi i} \int_{Ha_-} u^{-\nu - 1} \exp\left[u - z^2/(4u)\right] du\,, \qquad (B.58)$$

where the contour is unchanged since z is real. But the integral in (B.58) defines an entire function of z because it is single-valued and absolutely convergent for all z. We recognize from the pre-factor in (B.58) that $J_\nu(z)$ has a branch point in the origin, if ν is not integer.

In this case, according to the usual convention, we must introduce a branch cut on the negative real axis so that Eq. (B.58) is valid under the restriction $-\pi < \arg(z) < \pi$.

We note that the series expansion of $J_\nu(z)$, Eq. (B.1) may be obtained from the integral representation (B.58) by replacing $\exp[-z^2/(4u)]$ by its Taylor series and integrating term by term and finally using Hankel's integral representation of the reciprocal of the Gamma function, Eq. (A.19a). Of course, the procedure can be inverted to yield the integral representation (B.58) from the series representation (B.1).

Other integral representations related to the class of Bessel functions can be found in any handbook of special functions.

Laplace transform pairs. Herewith we report a few Laplace transform pairs related to Bessel functions extracted from [Ghizzetti and Ossicini (1971)], where the interested reader can find more formulas, all with the proof included. We first consider

$$J_\nu(\alpha t) \div \frac{\left(\sqrt{s^2+\alpha^2}-s\right)^\nu}{\alpha^\nu\sqrt{s^2+\alpha^2}}\,,\ \mathcal{R}e\,\nu > -1\,,\ \mathcal{R}e\,s > |\mathcal{I}m\,\alpha|\,, \quad (B.59)$$

$$I_\nu(\alpha t) \div \frac{\left(s-\sqrt{s^2-\alpha^2}\right)^\nu}{\alpha^\nu\sqrt{s^2-\alpha^2}}\,,\ \mathcal{R}e\,\nu > -1\,,\ \mathcal{R}e\,s > |\mathcal{R}e\,\alpha|\,. \quad (B.60)$$

Then, we consider the following transform pairs relevant for wave propagation problems:

$$J_0\left(\alpha\sqrt{t^2-a^2}\right)\Theta(t-a) \div \frac{\mathrm{e}^{-a\sqrt{s^2+\alpha^2}}}{\sqrt{s^2+\alpha^2}}\,,\ \mathcal{R}e\,s > |\mathcal{I}m\,\alpha|\,, \quad (B.61)$$

$$I_0\left(\alpha\sqrt{t^2-a^2}\right)\Theta(t-a) \div \frac{\mathrm{e}^{-a\sqrt{s^2-\alpha^2}}}{\sqrt{s^2-\alpha^2}}\,,\ \mathcal{R}e\,s > |\mathcal{R}e\,\alpha|\,, \quad (B.62)$$

$$a\alpha\,\frac{J_1\left(\alpha\sqrt{t^2-a^2}\right)}{\sqrt{t^2-a^2}}\,\Theta(t-a) \div \mathrm{e}^{-as}-\mathrm{e}^{-a\sqrt{s^2+\alpha^2}}\,, \quad (B.63)$$
$$\mathcal{R}e\,s > |\mathcal{I}m\,\alpha|,$$

$$a\alpha\,\frac{I_1\left(\alpha\sqrt{t^2-a^2}\right)}{\sqrt{t^2-a^2}}\,\Theta(t-a) \div \mathrm{e}^{-as}-\mathrm{e}^{-a\sqrt{s^2-\alpha^2}}\,, \quad (B.64)$$
$$\mathcal{R}e\,s > |\mathcal{R}e\,\alpha|\,.$$

B.4 The Airy functions

The Airy differential equation in the complex plane ($z \in \mathbb{C}$). The Airy functions $Ai(z)$, $Bi(z)$ are usually introduced as the two linear independent solutions of the differential equation

$$\frac{d^2}{dz^2}u(z) - z\,u(z) = 0\,. \qquad (B.65)$$

The Wronskian is

$$W\{Ai(z), Bi(z)\} = \frac{1}{\pi}\,. \qquad B(65)$$

Taylor series.

$$\begin{aligned} Ai(z) = 3^{-2/3}&\sum_{n=0}^{\infty} \frac{z^{3n}}{9\,n!\,\Gamma(n+2/3)} \\ -3^{-4/3}&\sum_{n=0}^{\infty} \frac{z^{3n+1}}{9\,n!\,\Gamma(n+4/3)}\,. \end{aligned} \qquad (B.66a)$$

$$\begin{aligned} Bi(z) = 3^{-1/6}&\sum_{n=0}^{\infty} \frac{z^{3n}}{9\,n!\,\Gamma(n+2/3)} \\ +3^{-5/6}&\sum_{n=0}^{\infty} \frac{z^{3n+1}}{9\,n!\,\Gamma(n+4/3)}\,. \end{aligned} \qquad (B.66b)$$

We note

$$Ai(0) = Bi(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3) \approx 0.355\,. \qquad (B.67)$$

Functional relations.

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0\,,\ \omega = \mathrm{e}^{-2i\pi/3}\,. \qquad (B.68a)$$

$$Bi(z) = i\omega Ai(\omega z) - i\omega^2 Ai(\omega^2 z)\,,\ \omega = \mathrm{e}^{-2i\pi/3}\,. \qquad (B.68b)$$

Relations with Bessel functions.

$$\begin{cases} Ai(z) = \frac{1}{3} z^{1/2}\left[I_{-1/3}(\zeta) - I_{1/3}(\zeta)\right]\,, \\ Ai(-z) = \frac{1}{3} z^{1/2}\left[J_{1/3}(\zeta) + J_{-1/3}(\zeta)\right]\,, \end{cases} \qquad \zeta = \frac{2}{3} z^{3/2}\,. \qquad (B.69a)$$

$$\begin{cases} Bi(z) = \frac{1}{\sqrt{3}} z^{1/2}\left[I_{-1/3}(\zeta) + I_{1/3}(\zeta)\right]\,, \\ Bi(-z) = \frac{1}{\sqrt{3}} z^{1/2}\left[J_{-1/3}(\zeta) - J_{1/3}(\zeta)\right]\,, \end{cases} \qquad \zeta = \frac{2}{3} z^{3/2}\,. \qquad (B.69b)$$

Asymptotic representations.

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \mathrm{e}^{-2z^{3/2}/3}, \quad z \to \infty, \quad |\arg z| < \pi. \quad (B.70a)$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \mathrm{e}^{2z^{3/2}/3}, \quad z \to \infty, \quad |\arg z| < \pi/3. \quad (B.70a)$$

Integral representations for real variable ($z = x \in \mathbb{R}$).

$$\begin{aligned} Ai(x) &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{e}^{\zeta x - \zeta^3/3}\, d\zeta \\ &= \frac{1}{\pi} \int_0^{\infty} \cos\left(ux + u^3/3\right)\, du\,. \end{aligned} \quad (B.71a)$$

$$Bi(x) = \frac{1}{\pi} \int_0^{\infty} \left[\mathrm{e}^{ux - u^3/3} + \sin\left(ux + u^3/3\right) \right] du\,. \quad (B.71b)$$

Asymptotic representations for real variable ($z = x \in \mathbb{R}$).

$$Ai(x) \sim \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-2x^{3/2}/3\right), & x \to +\infty, \\ \frac{1}{\sqrt{\pi}} |x|^{-1/4} \sin\left(|x|^{3/2}/3 + \pi/4\right), & x \to -\infty. \end{cases} \quad (B.72a)$$

$$Bi(x) \sim \begin{cases} \frac{1}{\sqrt{\pi}} x^{-1/4} \exp\left(2x^{3/2}/3\right), & x \to +\infty, \\ \frac{1}{\sqrt{\pi}} |x|^{-1/4} \cos\left(|x|^{3/2}/3 + \pi/4\right), & x \to -\infty. \end{cases} \quad (B.72b)$$

Graphical representations for real variable ($z = x \in \mathbb{R}$). We present the plots of the Airy functions with their derivatives on the real line in Figs B.5 and B.6.

As expected from their relations with the Bessel functions, see Eqs. (B.69a) and (B.69b), and from their asymptotic representations, see Eqs (B.72a), (B.72b), we note from the plots that for $x > 0$ the functions $Ai(x)$, $Bi(x)$ are monotonic ($Ai(x)$ is exponentially decreasing, $Bi(x)$ is exponentially increasing), whereas for $x < 0$ both of them are oscillating with a slowly diminishing period and an amplitude decaying as $|x|^{-1/4}$. These changes in behaviour along the real line are the most noteworthy characteristics of the Airy functions.

For a survey on the applications of the Airy functions in physics we refer the interested reader to [Vallé and Soares (2004)].

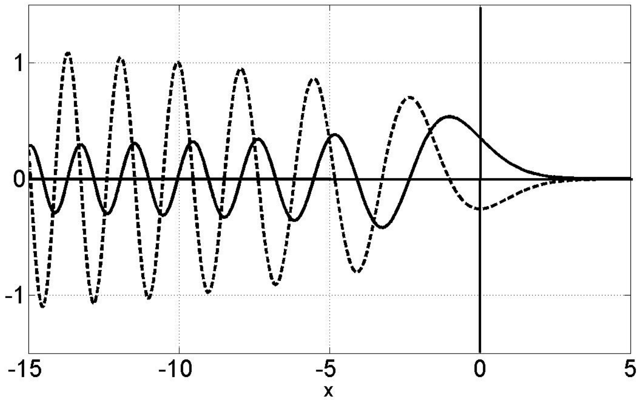

Fig. B.5 Plots of $Ai(x)$ (continuous line) and its derivative $Ai'(x)$ (dotted line) for $-15 \leq x \leq 5$.

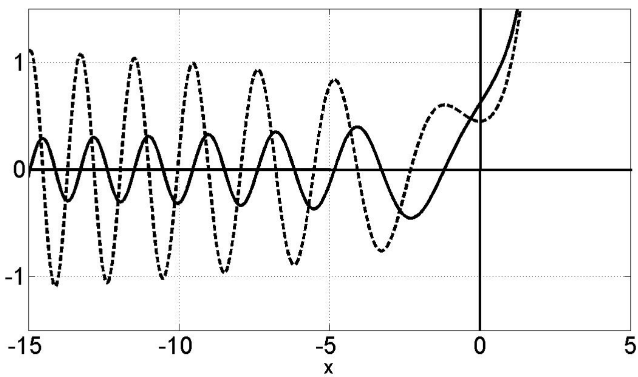

Fig. B.6 Plots of $Bi(x)$ (continuous line) and its derivative $Bi'(x)$ (dotted line) for $-15 \leq x \leq 5$.