Appendix F

The Wright Functions

In this appendix we provide a survey of the high transcendental functions known in the literature as Wright functions. We devote particular attention for two functions of the Wright type, which, in virtue of their role in applications of fractional calculus, we have called auxiliary functions. We also discuss their relevance in probability theory showing their connections with Lévy stable distributions. At the end, we add some historical and bibliographical notes.

F.1 The Wright function $W_{\lambda,\mu}(z)$

The *Wright function*, that we denote by $W_{\lambda,\mu}(z)$, is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the theory of partitions, see [Wright (1933); (1935a); (1935b)]. The function is defined by the series representation, convergent in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,, \tag{F.1}$$

so $W_{\lambda,\mu}(z)$ is an *entire function*. Originally, Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$, see [Wright (1940)]. We note that in the handbook of the Bateman Project [Erdélyi *et al.* (1953-1955)], Vol. 3, Ch. 18, presumably for a misprint, λ is restricted to be non-negative. We distinguish the Wright functions in *first kind* ($\lambda \geq 0$) and *second kind* ($-1 < \lambda < 0$).

The integral representation. The *integral representation* reads

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}}, \quad \lambda > -1, \quad \mu \in \mathbb{C}, \qquad (F.2)$$

where Ha denotes the Hankel path. The equivalence between the series and integral representations is easily proven by using the Hankel formula for the Gamma function, see (A.19),

$$\frac{1}{\Gamma(\zeta)} = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{u}\, u^{-\zeta}\, du, \quad \zeta \in \mathbb{C},$$

and performing a term-by-term integration. The exchange between series and integral is legitimate by the uniform convergence of the series, being $W_{\lambda,\mu(z)}$ an entire function. We have:

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}} = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma} \left[\sum_{n=0}^{\infty} \frac{z^n}{n!}\, \sigma^{-\lambda n}\right] \frac{d\sigma}{\sigma^{\mu}}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[\frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma}\, \sigma^{-\lambda n - \mu}\, d\sigma\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!\, \Gamma[\lambda n + \mu]}\,.$$

Furthermore, it is possible to prove that the Wright function is entire of order $1/(1+\lambda)$ hence of exponential type only if $\lambda \geq 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = \mathrm{e}^{z}/\Gamma(\mu)$.

Asymptotic expansions. For the detailed asymptotic analysis in the whole complex plane for the Wright functions, the interested reader is referred to [Wong and Zhao (1999a); (1999b)]. These authors have provided asymptotic expansions of the Wright functions of the first and second kind following a new method for smoothing Stokes' discontinuities.

As a matter of fact, the second kind is the most interesting for us. By setting $\lambda = -\nu \in (-1, 0)$, we recall the asymptotic expansion originally obtained by Wright himself, that is valid in a suitable sector about the negative real axis as $|z| \to \infty$,

$$W_{-\nu,\mu}(z) = Y^{\,1/2-\mu}\, \mathrm{e}^{-Y} \left[\sum_{m=0}^{M-1} A_m\, Y^{-m} + O(|Y|^{-M})\right], \qquad (F.3)$$
$$Y = Y(z) = (1-\nu)\,(-\nu^{\nu}\, z)^{1/(1-\nu)},$$

where the A_m are certain real numbers.

Generalization of the Bessel functions. The Wright functions turn out to be related to the well-known Bessel functions J_ν and I_ν for $\lambda = 1$ and $\mu = \nu + 1$. In fact, by using the series definitions (B.1) and (B.31) for the Bessel functions and the series definitions (F.1) for the Wright functions, we easily recognize the identities:

$$\begin{aligned} J_\nu(z) &:= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n + +\nu + 1)} = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{z^2}{4}\right), \\ W_{1,\nu+1}(-z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma(n + \nu + 1)} = z^{-\nu/2}\, J_\nu(2z^{1/2}). \end{aligned} \qquad (F.4)$$

and

$$\begin{aligned} I_\nu(z) &:= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n + +\nu + 1)} = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{z^2}{4}\right), \\ W_{1,\nu+1}(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(n + \nu + 1)} = z^{-\nu/2}\, I_\nu(2z^{1/2}). \end{aligned} \qquad (F.5)$$

As far as the standard Bessel functions J_ν are concerned, the following observations are worth noting. We first note that the Wright function $W_{1,\nu+1}(-z)$ reduces to the entire function $\mathcal{C}_\nu(z)$ known as *Bessel-Clifford function* introduced Eq. (B.4). Then, in view of the first equation in (F.4) some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel-Maitland function*) and introduce the notation for $\lambda \geq 0$, see e.g. [Kiryakova (1994)], p. 336,

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(\lambda n + \nu + 1)} = \left(\frac{z}{2}\right)^\nu W_{\lambda,\nu+1}\left(-\frac{z^2}{4}\right). \qquad (F.6)$$

Similar remarks can be extended to the modified Bessel functions I_ν.

Recurrence relations. Hereafter, we quote some relevant recurrence relations from [Erdélyi *et al.* (1953-1954)], Vol. 3, Ch. 18:

$$\lambda z\, W_{\lambda,\lambda+\mu}(z) = W_{\lambda,\mu-1}(z) + (1-\mu)\, W_{\lambda,\mu}(z)\,, \qquad (F.7)$$

$$\frac{d}{dz} W_{\lambda,\mu}(z) = W_{\lambda,\lambda+\mu}(z)\,. \qquad (F.8)$$

We note that these relations can easily be derived from (F.1).

F.2 The auxiliary functions $F_\nu(z)$ and $M_\nu(z)$ in $\mathbb{C}$

In his earliest analysis of the time-fractional diffusion-wave equation [Mainardi (1994a)], the author introduced the two *auxiliary functions* of the Wright type:

$$F_\nu(z) := W_{-\nu,0}(-z)\,, \quad 0 < \nu < 1\,, \tag{F.9}$$

and

$$M_\nu(z) := W_{-\nu,1-\nu}(-z)\,, \quad 0 < \nu < 1\,, \tag{F.10}$$

interrelated through

$$F_\nu(z) = \nu\, z\, M_\nu(z)\,. \tag{F.11}$$

As it is shown in Chapter 6, the motivation was based on the inversion of certain Laplace transforms in order to obtain the fundamental solutions of the fractional diffusion-wave equation in the space-time domain. Here we will devote particular attention to the mathematical properties of these functions limiting at the essential the discussion for the general Wright functions. The reader is referred to the Notes for some historical and bibliographical details.

Series representations. The *series representations* of our auxiliary functions are derived from those of $W_{\lambda,\mu}(z)$. We have:

$$\begin{aligned} F_\nu(z) &:= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\nu n)} \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!}\,\Gamma(\nu n+1)\,\sin(\pi\nu n)\,, \end{aligned} \tag{F.12}$$

and

$$\begin{aligned} M_\nu(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n+(1-\nu)]} \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\,\Gamma(\nu n)\,\sin(\pi\nu n)\,, \end{aligned} \tag{F.13}$$

where we have used the well-known reflection formula for the Gamma function, see (A.13),

$$\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi/\sin \pi\zeta\,.$$

We note that $F_\nu(0) = 0\,,\ M_\nu(0) = 1/\Gamma(1-\nu)$ and that the relation (F.11), consistent with the recurrence relation (F.7), can be derived from (F.12)-(F.13) arranging the terms of the series.

The integral representations. The *integral representations* of our auxiliary functions are derived from those of $W_{\lambda,\mu}(z)$. We have:

$$F_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma\, -\, z\sigma^\nu}\, d\sigma\,, \quad z \in \mathbb{C}\,, \quad 0 < \nu < 1\,, \qquad (F.14)$$

$$M_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma\, -\, z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}}\,, \quad z \in \mathbb{C}\,, \quad 0 < \nu < 1\,. \qquad (F.15)$$

We note that the relation (F.11) can be obtained directly from (F.14) and (F.15) with an integration by parts, i.e.

$$\int_{Ha} \mathrm{e}^{\,\sigma\, -\, z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}} = \int_{Ha} \mathrm{e}^{\,\sigma}\left(-\frac{1}{\nu z}\,\frac{d}{d\sigma}\mathrm{e}^{\,-z\sigma^\nu}\right) d\sigma$$
$$= \frac{1}{\nu z}\int_{Ha} \mathrm{e}^{\,\sigma\, -\, z\sigma^\nu}\, d\sigma\,.$$

The passage from the series representation to the integral representation and vice-versa for our auxiliary functions can be derived in a way similar to that adopted for the general Wright function, that is by expanding in positive powers of z the exponential function $\exp(-z\,\sigma^\nu)$, exchanging the order between the series and the integral and using the Hankel representation of the reciprocal of the Gamma function, see (A.19a).

Since the radius of convergence of the power series in (F.12)-(F.13) can be proven to be infinite for $0 < \nu < 1$, our auxiliary functions turn out to be entire in z and therefore the exchange between the series and the integral is legitimate[7].

Special cases. Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of ν.

In [Mainardi and Tomirotti (1995)] the authors have shown that for $\nu = 1/q$, where $q \geq 2$ is a positive integer, the auxiliary functions can be expressed as a sum of $(q-1)$ simpler entire functions.

In the particular cases $q = 2$ and $q = 3$ we find from (F.13),

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\sum_{m=0}^{\infty}(-1)^m\left(\frac{1}{2}\right)_m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}}\exp\left(-\,z^2/4\right), \quad (F.16)$$

[7] The author in [Mainardi (1994a)] proved these properties independently from [Wright (1940)], because at that time he was aware only of [Erdélyi *et al.* (1953-1955)] where λ was restricted to be non-negative.

and

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)_m \frac{z^{3m}}{(3m)!} - \frac{1}{\Gamma(1/3)} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)_m \frac{z^{3m+1}}{(3m+1)!} \tag{F.17}$$

$$= 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right)\,,$$

where *Ai* denotes the *Airy function* defined in Appendix B (Section B.4).

Furthermore, it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q-1$

$$\frac{d^{q-1}}{dz^{q-1}}\, M_{1/q}(z) + \frac{(-1)^q}{q}\, z\, M_{1/q}(z) = 0\,, \tag{F.18}$$

subjected to the $q-1$ initial conditions at $z=0$, derived from (F.13),

$$M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi}\, \Gamma[(h+1)/q]\, \sin[\pi\,(h+1)/q]\,, \tag{F.19}$$

with $h = 0,\, 1,\, \ldots\, q-2$.

We note that, for $q \geq 4$, Eq. (F.18) is akin to the *hyper-Airy* differential equation of order $q-1$, see e.g. [Bender and Orszag (1987)]. Consequently, in view of the above considerations, the auxiliary function $M_\nu(z)$ could be referred to as the *generalized hyper-Airy function.*

F.3 The auxiliary functions $F_\nu(x)$ and $M_\nu(x)$ in $\mathbb{R}$

We point out that the most relevant applications of Wright functions, especially our auxiliary functions, are when the independent variable is real. More precisely, in this Section we will consider functions of the variable x with $x \in \mathbb{R}^+$ or $x \in \mathbb{R}$.

When the support is all of $\mathbb{R}$, we agree to consider *even functions*, that is, functions defined in a symmetric way. In this case, to stress the symmetry property of the function, the independent variable may be denoted by $|x|$.

We point out that in the limit $\nu \to 1^-$ the function $M_\nu(x)$, for $x \in \mathbb{R}^+$, tends to the Dirac generalized function $\delta(x-1)$.

The asymptotic representation of $M_\nu(x)$. Let us first point out the asymptotic behaviour of the function $M_\nu(x)$ as $x \to +\infty$. Choosing as a variable x/ν rather than x, the computation of the asymptotic representation by the saddle-point approximation yields, see [Mainardi and Tomirotti (1995)],

$$M_\nu(x/\nu) \sim a(\nu)\, x^{(\nu - 1/2)/(1-\nu)} \exp\left[-b(\nu)\, x^{1/(1-\nu)}\right], \quad (F.20)$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi\,(1-\nu)}} > 0\,, \quad b(\nu) = \frac{1-\nu}{\nu} > 0\,. \quad (F.21)$$

The above evaluation is consistent with the first term in Wright's asymptotic expansion (F.3) after having used the definition (F.10).

Plots of $M_\nu(x)$. We show the plots of our auxiliary functions on the real axis for some rational values of the parameter ν.

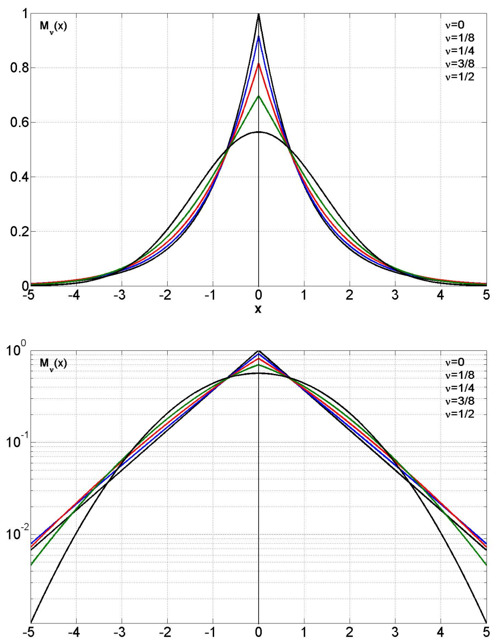

Fig. F.1 Plots of the Wright type function $M_\nu(x)$ with $\nu = 0, 1/8, 1/4, 3/8, 1/2$ for $-5 \le x \le 5$; top: linear scale, bottom: logarithmic scale.

To gain more insight of the effect of the parameter itself on the behaviour close to and far from the origin, we will adopt both linear and logarithmic scale for the ordinates.

In Figs. F.1 and F.2 we compare the plots of the $M_\nu(x)$ Wright auxiliary functions in $-5 \leq x \leq 5$ for some rational values in the ranges $\nu \in [0, 1/2]$ and $\nu \in [1/2, 1]$, respectively. Thus in Fig. F.1 we see the transition from $\exp(-|x|)$ for $\nu = 0$ to $1/\sqrt{\pi}\,\exp(-x^2)$ for $\nu = 1/2$, whereas in Fig F.2 we see the transition from $1/\sqrt{\pi}\,\exp(-x^2)$ to the delta function $\delta(1 - |x|)$ for $\nu = 1$.

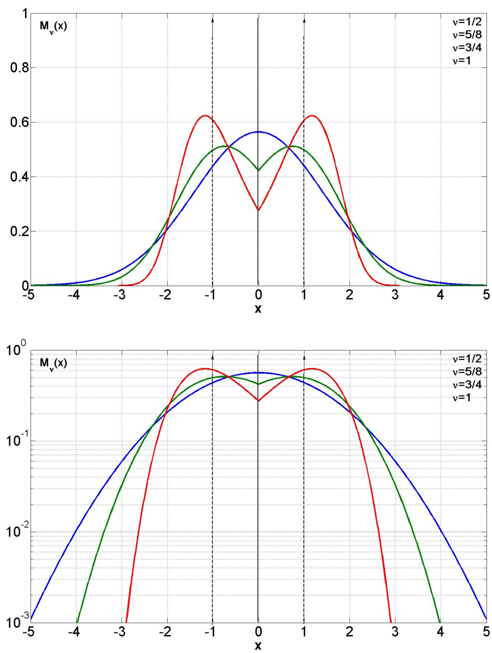

Fig. F.2 Plots of the Wright type function $M_\nu(x)$ with $\nu = 1/2\,,\ 5/8\,,\ 3/4\,,\ 1$ for $-5 \leq x \leq 5$: top: linear scale; bottom: logarithmic scale.

In plotting $M_\nu(x)$ at fixed ν for sufficiently large x the asymptotic representation (F.20)-(F.21) is very useful because, as x increases, the numerical convergence of the series in (F.13) becomes poor and poor up to being completely inefficient. Henceforth, the matching between the series and the asymptotic representation is relevant.

However, as $\nu \to 1^-$, the plotting remains a very difficult task because of the high peak arising around $x = \pm 1$. In this case the saddle-point method, improved as in [Kreis and Pipkin (1986)], can successfully be used to visualize some structure in the peak while it tends to the Dirac delta function, see also [Mainardi and Tomirotti (1997)] and Chapter 6 for a related wave-propagation problem. With Pipkin's method we are able to get the desired matching with the series representation just in the region around the maximum $x \approx 1$, as shown in Fig. F.3. Here we exhibit the significant plots of the auxiliary function $M_\nu(x)$ with $\nu = 1 - \epsilon$ for $\epsilon = 0.01$ and $\epsilon = 0.001$ and we compare the series representation (100 terms, dashed line), the saddle-point representation (dashed-dotted line), and the Pipkin representation (continuous line).

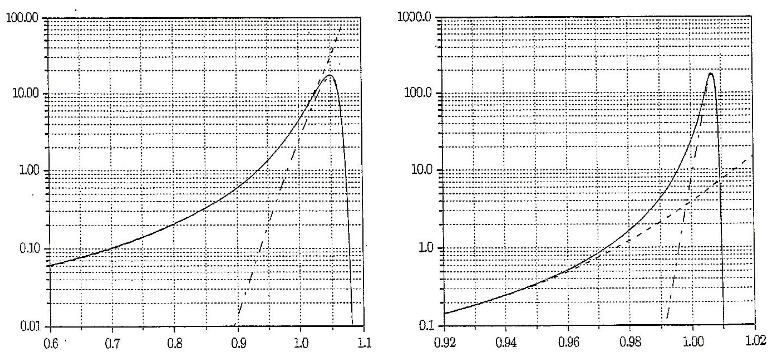

Fig. F.3 Comparison of the representations of $M_\nu(x)$ with $\nu = 1 - \epsilon$ around the maximum $x \approx 1$ obtained by Pipkin's method (continuous line), 100 terms-series (dashed line) and the saddle-point method (dashed-dotted line). Left:$\epsilon = 0.01$; Right: $\epsilon = 0.001$

F.4 The Laplace transform pairs

Let us write the Laplace transform of the Wright function as

$$W_{\lambda,\mu}(\pm r) \div \mathcal{L}\,[W_{\lambda,\mu}(\pm r); s] := \int_0^\infty \mathrm{e}^{-s\,r}\, W_{\lambda,\mu}(\pm r)\, dr\,,$$

where r denotes a non-negative real variable, i.e. $0 \le r < +\infty$, and s is the Laplace complex parameter.

When $\lambda > 0$ the series representation of the Wright function can be transformed term-by-term. In fact, for a known theorem of the theory of the Laplace transforms, see e.g. [Doetsch (1974)], the Laplace transform of an entire function of exponential type can be obtained by transforming term-by-term the Taylor expansion of the original function around the origin. In this case the resulting Laplace transform turns out to be analytic and vanishing at infinity. As a consequence, we obtain *the Laplace transform pair for the Wright function of the first kind* as

$$W_{\lambda,\mu}(\pm r) \div \frac{1}{s} E_{\lambda,\mu}\left(\pm\frac{1}{s}\right), \quad \lambda > 0, \quad |s| > \rho > 0, \qquad (F.22)$$

where $E_{\lambda,\mu}$ denotes the generalized Mittag-Leffler function in two parameters, and ρ is an arbitrary positive number. The proof is straightforward, noting that

$$\sum_{n=0}^{\infty} \frac{(\pm r)^n}{n!\,\Gamma(\lambda n+\mu)} \div \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\pm 1/s)^n}{\Gamma(\lambda n+\mu)},$$

and recalling the series representation (E.22) of the generalized Mittag-Leffler function,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

For $\lambda \to 0^+$ Eq. (F.22) provides the Laplace transform pair

$$W_{0^+,\mu}(\pm r) := \frac{\mathrm{e}^{\pm r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s \mp 1}.$$

This means

$$W_{0^+,\mu}(\pm r) \div \frac{1}{s} E_{0,\mu}\left(\pm\frac{1}{s}\right) = \frac{1}{\Gamma(\mu)s} E_0\left(\pm\frac{1}{s}\right), \; |s| > 1, \quad (F.23)$$

where, in order to be consistent with (F.22), we have formally put, according to (E.2),

$$E_{0,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu)} = \frac{1}{\Gamma(\mu)} E_0(z) = \frac{1}{\Gamma(\mu)} \frac{1}{1-z}, \quad |z| < 1.$$

We recognize that in this limitig case the Laplace transform exhibits a simple pole at $s = \pm 1$ while for $\lambda > 0$ it exhibits an essential singularity at $s = 0$.

For $-1 < \lambda < 0$ the Wright function turns out to be an entire function of order greater than 1, so that the term-by-term transformation representation is no longer legitimate. Thus, for Wright functions of the second kind, care is required in establishing the existence of the Laplace transform, which necessarily must tend to zero as $s \to \infty$ in its half-plane of convergence.

For the sake of convenience we first derive the Laplace transform for the special case of $M_\nu(r)$; the exponential decay as $r \to \infty$ of the *original* function provided by (F.20) ensures the existence of the *image* function. From the integral representation (F.13) of the M_ν function we obtain

$$M_\nu(r) \div \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s\,r} \left[\int_{Ha} \mathrm{e}^{\sigma - r\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} \right] dr$$

$$= \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma}\, \sigma^{\nu-1} \left[\int_0^\infty \mathrm{e}^{-r(s+\sigma^\nu)}\, dr \right] d\sigma = \frac{1}{2\pi i} \int_{Ha} \frac{\mathrm{e}^{\sigma}\, \sigma^{\nu-1}}{\sigma^\nu + s}\, d\sigma\,.$$

Then, by recalling the integral representation (E.14) of the Mittag-Leffler function,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha > 0\,,$$

we obtain the Laplace transform pair

$$M_\nu(r) \div E_\nu(-s)\,, \quad 0 < \nu < 1\,. \qquad (F.24)$$

Although transforming the Taylor series of $M_\nu(r)$ term-by-term is not legitimate, this procedure yields a series of negative powers of s that represents the asymptotic expansion of the correct Laplace transform, $E_\nu(-s)$, as $s \to \infty$ in a sector around the positive real axis. Indeed we get

$$\sum_{n=0}^{\infty} \frac{\int_0^\infty \mathrm{e}^{-sr}(-r)^n\, dr}{n!\Gamma(-\nu n + (1-\nu))} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(-\nu n + 1 - \nu)} \frac{1}{s^{n+1}}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\Gamma(-\nu m + 1)} \frac{1}{s^m} \sim E_\nu(-s)\,,\ s \to \infty\,.,$$

consistently with the asymptotic expansion (E.16).

We note that (F.24) contains the well-known Laplace transform pair, see e.g. [Doetsch (1974)],

$$M_{1/2}(r) := \frac{1}{\sqrt{\pi}} \exp\left(-\,r^2/4\right) \div E_{1/2}(-s) := \exp\left(s^2\right) \mathrm{erfc}(s)\,,$$

that is valid for all $s \in \mathbb{C}$.

Analogously, using the more general integral representation (F.2) of the Wright function, we can get *the Laplace transform pair for the Wright function of the second kind.* For the case $\lambda = -\nu \in (-1, 0)$, with $\mu > 0$ for simplicity, we obtain,

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s)\,, \quad 0 < \nu < 1\,. \qquad (F.25)$$

We note the minus sign in the argument in order to ensure the the existence of the Laplace transform thanks to the Wright asymptotic formula (F.3) valid in a sector about the negative real axis.

In the limit as $\lambda \to 0^-$ we formally obtain the Laplace transform pair

$$W_{0^-,\mu}(-r) := \frac{\mathrm{e}^{-r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)}\,\frac{1}{s+1}\,.$$

In order to be consistent with (F.24) we rewrite

$$W_{0^-,\mu}(-r) \div E_{0,\mu}(-s) = \frac{1}{\Gamma(\mu)}\,E_0(-s)\,,\ |s| < 1\,. \qquad (F.26)$$

Therefore, as $\lambda \to 0^{\pm}$, we note a sort of continuity in the formal results (F.23) and (F.26) because

$$\frac{1}{(s+1)} = \begin{cases} (1/s)\,E_0(-1/s)\,, & |s| > 1\,; \\ E_0(-s)\,, & |s| < 1\,. \end{cases} \qquad (F.27)$$

We now point out the relevant Laplace transform pair related to the *auxiliary* functions of argument $r^{-\nu}$ proved in [Mainardi (1994a; (1996a); (1996b)]:

$$\frac{1}{r}\,F_\nu\,(1/r^\nu) = \frac{\nu}{r^{\nu+1}}\,M_\nu\,(1/r^\nu) \div \mathrm{e}^{-s^\nu}\,, \quad 0 < \nu < 1\,. \qquad (F.28)$$

$$\frac{1}{\nu}\,F_\nu\,(1/r^\nu) = \frac{1}{r^\nu}\,M_\nu\,(1/r^\nu) \div \frac{\mathrm{e}^{-s^\nu}}{s^{1-\nu}}\,, \quad 0 < \nu < 1\,. \qquad (F.29)$$

We recall that the Laplace transform pairs in (F.28) were formerly considered by [Pollard (1946)], who provided a rigorous proof based on a formal result by [Humbert (1945)]. Later [Mikusiński (1959a)] achieved a similar result based on his theory of operational calculus, and finally, albeit unaware of the previous results, [Buchen and Mainardi (1975)] derived the result in a formal way, as stressed in

Chapter 5. We note, however, that all these authors were not informed about the Wright functions. To our actual knowledge, the former author who derived the Laplace transforms pairs (F.28)-(F.29) in terms of Wright functions of the second kind was [Stankovich (1970)].

Hereafter, we will provide two independent proofs of (F.28) by carrying out the inversion of $\exp(-s^\nu)$, either by the complex Bromwich integral formula, see [Mainardi (1994a); Mainardi (1996a)], or by the formal series method, see [Buchen and Mainardi (1975)]. Similarly, we can act for the Laplace transform pair (F.29).

For the complex integral approach we deform the Bromwich path Br into the Hankel path Ha, that is equivalent to the original path, and we set $\sigma = sr$. Recalling (F.14)-(F.15), we get

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,sr - s^\nu}\, ds = \frac{1}{2\pi i\, r}\int_{Ha} \mathrm{e}^{\,\sigma - (\sigma/r)^\nu}\, d\sigma$$
$$= \frac{1}{r}\, F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\, M_\nu\left(1/r^\nu\right)\,.$$

For the series approach, let us expand the Laplace transform in series of positive powers of s and formally invert term by term. Then, after recalling (F.12)-(F.13), we obtain:

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\mathcal{L}^{-1}\left[s^{\nu n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,\frac{r^{-\nu n-1}}{\Gamma(-\nu n)}$$
$$= \frac{1}{r}\, F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\, M_\nu\left(1/r^\nu\right)\,.$$

We note the relevance of Laplace transforms (F.24) and (F.28) in pointing out the non-negativity of the Wright function $M_\nu(x)$ and the complete monotonicity of the Mittag-Leffler functions $E_\nu(-x)$ for $x > 0$ and $0 < \nu < 1$. In fact, since $\exp\left(-s^\nu\right)$ denotes the Laplace transform of a probability density (precisely, the extremal Lévy stable density of index ν, see [Feller (1971)]) the L.H.S. of (F.28) must be non-negative, and so also must the L.H.S of F(24). As a matter of fact the Laplace transform pair (F.24) shows, replacing s by x, that the spectral representation of the Mittag-Leffler function $E_\nu(-x)$ is expressed in terms of the Wright M-function $M_\nu(r)$, that is:

$$E_\nu(-x) = \int_0^\infty \mathrm{e}^{\,-rx}\, M_\nu(r)\, dr\,,\ 0 < \nu < 1\,,\ x \geq 0\,. \qquad (F.30)$$

We now recognize that Eq. (F.30) is consistent with Eqs. (E.19)-(E.21) derived by [Pollard (1948)].

It is instructive to compare the spectral representation of $E_\nu(-x)$ with that of the function $E_\nu(-t^\nu)$. From Eqs. (E.56)-(E.57) we can write

$$E_\nu(-t^\nu) = \int_0^\infty \mathrm{e}^{-rt} K_\nu(r)\, dr\,, \quad 0<\nu<1\,,\ t\ge 0\,, \tag{F.31}$$

where the *spectral function* reads

$$K_\nu(r) = \frac{1}{\pi}\,\frac{r^{\nu-1}\,\sin(\nu\pi)}{r^{2\nu}+2\,r^\nu\,\cos(\nu\pi)+1}\,. \tag{F.32}$$

The relationship between $M_\nu(r)$ and $K_\nu(r)$ is worth exploring. Both functions are non-negative, integrable and normalized in $\mathbb{R}^+$, so they can be adopted in probability theory as density functions. The normalization conditions derive from Eqs (F.30) and (F.31) since

$$\int_0^{+\infty} M_\nu(r)\, dr = \int_0^{+\infty} K_\nu(r)\, dr = E_\nu(0) = 1\,.$$

In the following section we will discuss the probability interpretation of the M_ν function with support both in $\mathbb{R}^+$ and in $\mathbb{R}$ whereas for K_ν we note that it has been interpreted as spectral distribution of relaxation/retardation times in the fractional Zener viscoelastic model, see Chapter 3, Section 3.2, Fig. 3.3.

We also note that for certain renewal processes, functions of Mittag-Leffler and Wright type can be adopted as probability distributions of waiting times, as shown in [Mainardi *et al.* (2005)], where such distributions are compared. We refer the interested reader to that paper for details.

F.5 The Wright M-functions in probability

We have already recognized that the Wright M-function with support in $\mathbb{R}^+$ can be interpreted as probability density function (*pdf*). Consequently, extending the function in a symmetric way to all of $\mathbb{R}$ and dividing by 2 we have a *symmetric pdf* with support in $\mathbb{R}$. In the former case the variable is usually a time coordinate whereas in the latter the variable is the absolute value of a space coordinate. We now provide more details on these densities in the framework of the theory of probability. As in Section F.3, we agree to denote by x and $|x|$ the variables in $\mathbb{R}^+$ and $\mathbb{R}$, respectively.

The absolute moments of order $\boldsymbol{\delta}$. The *absolute moments* of order $\delta > -1$ of the Wright M-function in $\mathbb{R}^+$ are finite and turn out to be

$$\int_0^\infty x^\delta M_\nu(x)\, dx = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}\,, \quad \delta > -1\,, \quad 0 \le \nu < 1\,. \qquad (F.33)$$

In order to derive this fundamental result we proceed as follows, based on the integral representation (F.15).

$$\begin{aligned}\int_0^\infty x^\delta\, M_\nu(x)\, dx &= \int_0^\infty c^\delta \left[\frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma - x\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}}\right] dx \\ &= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma} \left[\int_0^\infty \mathrm{e}^{\,-x\sigma^\nu}\, x^\delta\, dx\right] \frac{d\sigma}{\sigma^{1-\nu}} \\ &= \frac{\Gamma(\delta+1)}{2\pi i}\int_{Ha} \frac{\mathrm{e}^\sigma}{\sigma^{\nu\delta+1}}\, d\sigma = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}\,.\end{aligned}$$

Above we have legitimate the exchange between the two integrals and we have used the identity

$$\int_0^\infty \mathrm{e}^{\,-x\sigma^\nu}\, x^\delta\, dx = \frac{\Gamma(\delta+1)}{(\sigma^\nu)^{\delta+1}}\,,$$

derived from (A.23) along with the Hankel formula (A.19a).

In particular, for $\delta = n \in \mathbb{N}$, the above formula provides the moments of integer order that can also be computed from the Laplace transform pair (F.24) as follows:

$$\int_0^{+\infty} x^n\, M_\nu(x)\, dx = \lim_{s\to 0}\,(-1)^n\, \frac{d^n}{ds^n}\, E_\nu(-s) = \frac{\Gamma(n+1)}{\Gamma(\nu n+1)}\,.$$

Incidentally, we note that the Laplace transform pair (F.24) could be obtained using the fundamental result (F.33) by developing in power series the exponential kernel of the Laplace transform and then transforming the series term-by-term.

The characteristic function. As well-known in probability theory the Fourier transform of a density provides the so-called *characteristic function.* In our case we have:

$$\begin{aligned}\mathcal{F}\left[\tfrac{1}{2}M_\nu(|x|)\right] &:= \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, M_\nu(|x|)\, dx \\ &= \int_0^\infty \cos(\kappa x)\, M_\nu(x)\, dx = E_{2\nu}(-\kappa^2)\,.\end{aligned} \tag{F.34}$$

For this prove it is sufficient to develop in series the cosine function and use formula (F.33),

$$\begin{aligned}\int_0^\infty \cos(\kappa x)\, M_\nu(x)\, dx &= \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n}}{(2n)!} \int_0^\infty x^{2n}\, M_\nu(x)\, dx \\ &= \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n}}{\Gamma(2\nu n+1)} = E_{2\nu}(-\kappa^2)\,.\end{aligned}$$

Relations with Lévy stable distributions. We find it worthwhile to discuss the relations between the Wright M-functions and the so-called *Lévy stable distributions.* The term stable has been assigned by the French mathematician Paul Lévy, who, in the tuenties of the last century, started a systematic research in order to generalize the celebrated *Central Limit Theorem* to probability distributions with infinite variance. For stable distributions we can assume the following DEFINITION: *If two independent real random variables with the same shape or type of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be stable.*

The restrictive condition of stability enabled Lévy (and then other authors) to derive the *canonic form* for the characteristic function of the densities of these distributions. Here we follow the parameterization in [Feller (1952);(1971)] revisited in [Gorenflo and Mainardi (1998b)] and in [Mainardi *et al.* (2001)]. Denoting by $L_\alpha^\theta(x)$ a generic stable density in $\mathbb{R}$, where α is the *index of stability* and and θ the asymmetry parameter, improperly called *skewness*, its characteristic function reads:

$$\begin{gathered}L_\alpha^\theta(x) \div \widehat{L}_\alpha^\theta(\kappa) = \exp\left[-\psi_\alpha^\theta(\kappa)\right], \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,, \\ 0 < \alpha \le 2\,,\ |\theta| \le \min\{\alpha, 2-\alpha\}\,.\end{gathered} \tag{F.35}$$

We note that the allowed region for the parameters α and θ turns out to be a diamond in the plane $\{\alpha, \theta\}$ with vertices in the points $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$, that we call the *Feller-Takayasu diamond*, see Fig. F.4. For values of θ on the border of the diamond (that is $\theta = \pm\alpha$ if $0 < \alpha < 1$, and $\theta = \pm(2-\alpha)$ if $1 < \alpha < 2$) we obtain the so-called *extremal stable densities*.

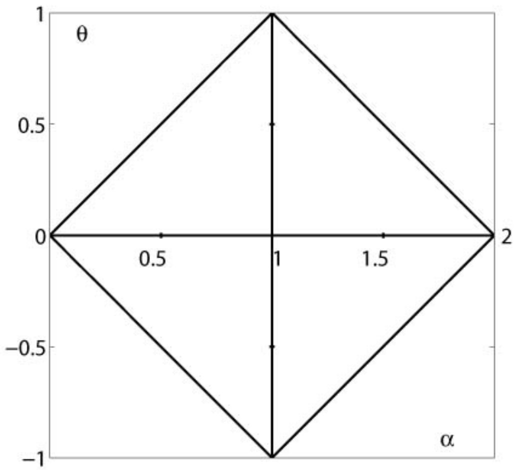

Fig. F.4 The Feller-Takayasu diamond for Lévy stable densities.

We note the *symmetry relation* $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$, so that a stable density with $\theta = 0$ is symmetric.

Stable distributions have noteworthy properties of which the interested reader can be informed from the relevant existing literature. Here-after we recall some peculiar PROPERTIES:

- *The class of stable distributions possesses its own domain of attraction*, see e.g. [Feller (1971)].
- *Any stable density is unimodal and indeed bell-shaped*, i.e. its n-th derivative has exactly n zeros in $\mathbb{R}$, see [Gawronski (1984)].
- *The stable distributions are self-similar and infinitely divisible.*

These properties derive from the canonic form (F.35) through the scaling property of the Fourier transform.

Self-similarity means

$$L_\alpha^\theta(x,t) \div \exp\left[-t\psi_\alpha^\theta(\kappa)\right] \Longleftrightarrow L_\alpha^\theta(x,t) = t^{-1/\alpha}\, L_\alpha^\theta(x/t^{1/\alpha})] \,, \quad (F.36)$$

where t is a positive parameter. If t is time, then $L_\alpha^\theta(x,t)$ is a spatial density evolving on time with self-similarity.
Infinite divisibility means that for every positive integer n, the characteristic function can be expressed as the nth power of some characteristic function, so that any stable distribution can be expressed as the n-fold convolution of a stable distribution of the same type. Indeed, taking in (F.35) $\theta = 0$, without loss of generality, we have

$$\mathrm{e}^{-t|\kappa|^\alpha} = \left[\mathrm{e}^{-(t/n)|\kappa|^\alpha}\right]^n \Longleftrightarrow L_\alpha^0(x,t) = \left[L_\alpha^0(x,t/n)\right]^{*n}, \qquad (F.37)$$

where

$$\left[L_\alpha^0(x,t/n)\right]^{*n} := L_\alpha^0(x,t/n) * L_\alpha^0(x,t/n) * \cdots * L_\alpha^0(x,t/n)$$

is the multiple Fourier convolution in $\mathbb{R}$ with n identical terms.

Only for a few particular cases, the inversion of the Fourier transform in (F.35) can be carried out using standard tables, and well-known probability distributions are obtained.

For $\alpha = 2$ (so $\theta = 0$), we recover the *Gaussian pdf*, that turns out to be the only stable density with finite variance, and more generally with finite moments of any order $\delta \geq 0$. In fact

$$L_2^0(x) = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-x^2/4}. \qquad (F.38)$$

All the other stable densities have finite absolute moments of order $\delta \in [-1,\alpha)$ as we will later show.

For $\alpha = 1$ and $|\theta| < 1$, we get

$$L_1^\theta(x) = \frac{1}{\pi}\frac{\cos(\theta\pi/2)}{[x+\sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2}, \qquad (F.39)$$

which for $\theta = 0$ includes the *Cauchy-Lorentz pdf*,

$$L_1^0(x) = \frac{1}{\pi}\frac{1}{1+x^2}. \qquad (F.40)$$

In the limiting cases $\theta = \pm 1$ for $\alpha = 1$ we obtain the *singular Dirac pdf's*

$$L_1^{\pm 1}(x) = \delta(x \pm 1). \qquad (F.41)$$

In general, we must recall the power series expansions provided in [Feller (1971)]. We restrict our attention to $x > 0$ since the evaluations for $x < 0$ can be obtained using the symmetry relation. The convergent expansions of $L_\alpha^\theta(x)$ ($x > 0$) turn out to be;

for $0 < \alpha < 1\,, \quad |\theta| \le \alpha$:

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \, \frac{\Gamma(1+n\alpha)}{n!} \, \sin\left[\frac{n\pi}{2}(\theta-\alpha)\right]\,; \qquad (F.42)$$

for $1 < \alpha \le 2\,, \quad |\theta| \le 2-\alpha$:

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x)^n \, \frac{\Gamma(1+n/\alpha)}{n!} \, \sin\left[\frac{n\pi}{2\alpha}(\theta-\alpha)\right]\,. \qquad (F.43)$$

From the series in (F.42) and the symmetry relation we note that *the extremal stable densities for* $0 < \alpha < 1$ *are unilateral*, precisely vanishing for $x > 0$ if $\theta = \alpha$, vanishing for $x < 0$ if $\theta = -\alpha$. In particular the unilateral extremal densities $L_\alpha^{-\alpha}(x)$ with $0 < \alpha < 1$ have support in $\mathbb{R}^+$ and Laplace transform $\exp(-s^\alpha)$. For $\alpha = 1/2$ we obtain the so-called *Lévy-Smirnov pdf*:

$$L_{1/2}^{-1/2}(x) = \frac{x^{-3/2}}{2\sqrt{\pi}} \, \mathrm{e}^{\,-1/(4x)}\,, \quad x \ge 0\,. \qquad (F.44)$$

As a consequence of the convergence of the series in (F.42)-(F.43) and of the symmetry relation we recognize that the stable *pdf*'s with $1 < \alpha \le 2$ are entire functions, whereas with $0 < \alpha < 1$ have the form

$$L_\alpha^\theta(x) = \begin{cases} (1/x)\, \Phi_1(x^{-\alpha}) & \text{for } x > 0\,, \\ (1/|x|)\, \Phi_2(|x|^{-\alpha}) & \text{for } x < 0\,, \end{cases} \qquad (F.45)$$

where $\Phi_1(z)$ and $\Phi_2(z)$ are distinct entire functions.The case $\alpha = 1$ ($|\theta| < 1$) must be considered in the limit for $\alpha \to 1$ of (F.42)-(F.43), because the corresponding series reduce to power series akin with geometric series in $1/x$ and x, respectively, with a finite radius of convergence. The corresponding stable *pdf*'s are no longer represented by entire functions, as can be noted directly from their explicit expressions (F.39)-(F.40).

We omit to provide the asymptotic representations of the stable densities referring the interested reader to [Mainardi *et al.* (2001)]. However, based on asymptotic representations, we can state as follows; for $0 < \alpha < 2$ the stable *pdf*'s exhibit *fat tails* in such a way that their absolute moment of order δ is finite only if $-1 < \delta < \alpha$.

More precisely, one can show that for non-Gaussian, not extremal, stable densities the asymptotic decay of the tails is

$$L_\alpha^\theta(x) = O\left(|x|^{-(\alpha+1)}\right) , \quad x \to \pm\infty \, . \tag{F.46}$$

For the extremal densities with $\alpha \neq 1$ this is valid only for one tail (as $|x| \to \infty$), the other (as $|x| \to \infty$) being of exponential order. For $1 < \alpha < 2$ the extremal *pdf*'s are two-sided and exhibit an exponential left tail (as $x \to -\infty$) if $\theta = +(2-\alpha)$, or an exponential right tail (as $x \to +\infty$) if $\theta = -(2-\alpha)$. Consequently, the Gaussian *pdf* is the unique stable density with finite variance. Furthermore, when $0 < \alpha \le 1$, the first absolute moment is infinite so we should use the median instead of the non-existent expected value in order to characterize the corresponding *pdf*.

Let us also recall a relevant identity between stable densities with index α and $1/\alpha$ (a sort of reciprocity relation) pointed out in [Feller (1971)], that is, assuming $x > 0$,

$$\frac{1}{x^{\alpha+1}} \, L_{1/\alpha}^{\theta}(x^{-\alpha}) = L_\alpha^{\theta^*}(x) \, , \; 1/2 \le \alpha \le 1 \, , \; \theta^* = \alpha(\theta+1) - 1 \, . \tag{F.47}$$

The condition $1/2 \le \alpha \le 1$ implies $1 \le 1/\alpha \le 2$. A check shows that θ^* falls within the prescribed range $|\theta^*| \le \alpha$ if $|\theta| \le 2 - 1/\alpha$. We leave as an exercise for the interested reader the verification of this reciprocity relation in the limiting cases $\alpha = 1/2$ and $\alpha = 1$.

From a comparison between the series expansions in (F.42)-(F.43) and in (F.12)-(F.13), we recognize that for $x > 0$ our *auxiliary functions of the Wright type are related to the extremal stable densities as follows*, see [Mainardi and Tomirotti (1997)],

$$L_\alpha^{-\alpha}(x) = \frac{1}{x} \, F_\alpha(x^{-\alpha}) = \frac{\alpha}{x^{\alpha+1}} \, M_\alpha(x^{-\alpha}) \, , \quad 0 < \alpha < 1 \, , \tag{F.48}$$

$$L_\alpha^{\alpha-2}(x) = \frac{1}{x} \, F_{1/\alpha}(x) = \frac{1}{\alpha} \, M_{1/\alpha}(x) \, , \quad 1 < \alpha \le 2 \, . \tag{F.49}$$

In Eqs. (F.48)-(F.49), for $\alpha = 1$, the skewness parameter turns out to be $\theta = -1$, so we get the singular limit

$$L_1^{-1}(x) = M_1(x) = \delta(x-1) \, . \tag{F.50}$$

More generally, all (regular) stable densities, given in Eqs. (F.42)-(F.43), were recognized to belong to the class of Fox H-functions, as formerly shown by [Schneider (1986)], see also [Mainardi *et al.* (2005)]. This general class of high transcendental functions is out of the scope of this book.

The Wright $\mathbb{M}$-function in two variables. In view of time-fractional diffusion processes related to time-fractional diffusion equations it is worthwhile to introduce the function in two variables

$$\mathbb{M}_\nu(x,t) := t^{-\nu}\, M_\nu(xt^{-\nu})\,, \quad 0<\nu<1\,, \quad x,t \in \mathbb{R}^+\,, \qquad (F.51)$$

which defines a spatial probability density in x evolving in time t with self-similarity exponent $H=\nu$. Of course for $x \in \mathbb{R}$ we have to consider the symmetric version obtained from (F.51) multiplying by $1/2$ and replacing x by $|x|$.

Hereafter we provide a list of the main properties of this function, which can be derived from the Laplace and Fourier transforms for the corresponding Wright M-function in one variable.

From Eq. (F.29) we derive the Laplace transform of $\mathbb{M}_\nu(x,t)$ with respect to $t \in \mathbb{R}^+$,

$$\mathcal{L}\left\{\mathbb{M}_\nu(x,t); t \to s\right\} = s^{\nu-1}\, \mathrm{e}^{\,-xs^\nu}\,. \qquad (F.52)$$

From Eq. (F.24) we derive the Laplace transform of $\mathbb{M}_\nu(x,t)$ with respect to $x \in \mathbb{R}^+$,

$$\mathcal{L}\left\{\mathbb{M}_\nu(x,t); x \to s\right\} = E_\nu\left(-st^\nu\right)\,. \qquad (F.53)$$

From Eq. (F.34) we derive the Fourier transform of $\mathbb{M}_\nu(|x|,t)$ with respect to $x \in \mathbb{R}$,

$$\mathcal{F}\left\{\mathbb{M}_\nu(|x|,t); x \to \kappa\right\} = 2E_{2\nu}\left(-\kappa^2 t^\nu\right)\,. \qquad (F.54)$$

Using the Mellin transforms [Mainardi *et al.* (2003)] derived the following integral formula,

$$\mathbb{M}_\nu(x,t) = \int_0^\infty \mathbb{M}_\lambda(x,\tau)\, \mathbb{M}_\mu(\tau,t)\, d\tau\,, \quad \nu = \lambda\mu\,. \qquad (F.55)$$

Special cases of the Wright $\mathbb{M}$-function are simply derived for $\nu = 1/2$ and $\nu = 1/3$ from the corresponding ones in the complex domain, see Eqs. (F.16)-(F.17). We devote particular attention to the case $\nu = 1/2$ for which we get from (F.16) the Gaussian density in $\mathbb{R}$,

$$\mathbb{M}_{1/2}(|x|,t) = \frac{1}{2\sqrt{\pi}t^{1/2}}\, \mathrm{e}^{\,-x^2/(4t)}\,. \qquad (F.56)$$

For the limiting case $\nu = 1$ we obtain

$$\mathbb{M}_1(|x|,t) = \frac{1}{2}\left[\delta(x-t) + \delta(x+t)\right]\,. \qquad (F.57)$$

F.6 Notes

In the early nineties, in his former analysis of fractional equations interpolating diffusion and wave-propagation, the present author, see e.g. [Mainardi (1994a)], introduced the functions of the Wright type $F_\nu(z) := W_{-\nu,0}(-z)$ and $M_\nu(z) := W_{-\nu,1-\nu}(-z)$ with $0 < \nu < 1$, in order to characterize the fundamental solutions for typical boundary value problems, as it is shown in Chapter 6.

Being then only aware of the Handbook of the Bateman project, where the parameter λ of the Wright function $W_{\lambda,\mu}(z)$ was erroneously restricted to non-negative values, the author thought to have originally extended the analyticity property of the original Wright function by taking $\nu = -\lambda$ with $\nu \in (0,1)$. So he introduced the entire functions F_ν and M_ν as *auxiliary functions* for his purposes. Presumably for this reason, the function M_ν is referred to as the *Mainardi function* in the treatise by [Podlubny (1999)] and in some research papers including [Balescu (2007a)], [Chechkin *et al.* (2008)], [Germano *et al.* (2009)], [Gorenflo *et al.* (1999); (2000)], [Hanyga (2002b)], [Kiryakova (2009a); (2009b)].

It was Professor B. Stanković, during the presentation of the paper [Mainardi and Tomirotti (1995)] at the Conference *Transform Methods and Special Functions, Sofia 1994*, who informed the author that this extension for $-1 < \lambda < 0$ had been already made by Wright himself in 1940 (following his previous papers in the thirties), see [Wright (1940)]. In his paper [Mainardi *et al.* (2005)], devoted to the 80th birthday of Professor Stanković, the author used the occasion to renew his personal gratitude to Professor Stanković for this earlier information that led him to study the original papers by Wright and to work (also in collaboration) on the functions of the Wright type for further applications.

For more mathematical details on the functions of the Wright type, the reader may be referred to [Kilbas *et al.* (2002)] and the references therein. For the numerical point of view we like to highlight the recent paper by [Luchko (2008)], where algorithms are provided for computation of the Wright function on the real axis with prescribed accuracy.

Furthermore, from the stochastic point of view, the Wright M-function emerges as a natural generalization of the Gaussian density for time-fractional diffusion processes. In fact, when these self-similar non-Markovian processes are characterized by stationary increments, so that they are defined only through their first and second moments, which indeed is a property of Gaussian processes, the Wright $M-pdf$ plays the main role as the Gaussian. Thus, such a class of processes, denoted as *generalized grey Brownian motion*, generalizes the Gaussian class of the *fractional Brownian motion* and covers stochastic models of anomalous diffusion, both of slow and fast type. See for details [Mura and Mainardi (2008)], [Mura and Pagnini (2008)], [Mura *et al.* (2008)] and the recent tutorial survey by [Mainardi *et al.* (2010)].