Preface

The aim of this monograph is essentially to investigate the connections among *fractional calculus*, *linear viscoelasticity* and *wave motion*. The treatment mainly reflects the research activity and style of the author in the related scientific areas during the last decades.

Fractional calculus, in allowing integrals and derivatives of any positive order (the term "fractional" is kept only for historical reasons), can be considered a branch of mathematical physics which deals with integro-differential equations, where integrals are of convolution type and exhibit weakly singular kernels of power law type.

Viscoelasticity is a property possessed by bodies which, when deformed, exhibit both viscous and elastic behaviour through simultaneous dissipation and storage of mechanical energy. It is known that viscosity refers mainly to fluids and elasticity mainly to solids, so we shall refer viscoelasticity to generic continuous media in the framework of a *linear* theory. As a matter of fact the linear theory of viscoelasticity seems to be the field where we find the most extensive applications of fractional calculus for a long time, even if often in an implicit way.

Wave motion is a wonderful world impossible to be precisely defined in a few words, so it is preferable to be guided in an intuitive way, as G.B. Whitham has pointed out. Wave motion is surely one of the most interesting and broadest scientific subjects that can be studied at any technical level. The restriction of wave propagation to linear viscoelastic media does not diminish the importance of this research area from mathematical and physical view points.

This book intends to show how fractional calculus provides a suitable (even if often empirical) method of describing dynamical properties of linear viscoelastic media including problems of wave propagation and diffusion. In all the applications the special transcendental functions are fundamental, in particular those of Mittag-Leffler and Wright type.

Here mathematics is emphasized for its own sake, but in the sense of a language for everyday use rather than as a body of theorems and proofs: unnecessary mathematical formalities are thus avoided. Emphasis is on problems and their solutions rather than on theorems and their proofs. So as not to bore a "practical" reader with too many mathematical details and functional spaces, we often skim over the regularity conditions that ensure the validity of the equations. A "rigorous" reader will be able to recognize these conditions, whereas a "practionist" reader will accept the equations for sufficiently well-behaved functions. Furthermore, for simplicity, the discussion is restricted to the scalar cases, i.e. one-dimensional problems.

The book is likely to be of interest to applied scientists and engineers. The presentation is intended to be self-contained but the level adopted supposes previous experience with the elementary aspects of mathematical analysis including the theory of integral transforms of Laplace and Fourier type.

By referring the reader to a number of appendices where some special functions used in the text are dealt with detail, the author intends to emphasize the mathematical and graphical aspects related to these functions.

Only seldom does the main text give references to the literature, the references are mainly deferred to notes sections at the end of chapters and appendices. The notes also provide some historical perspectives. The bibliography contains a remarkably large number of references to articles and books not mentioned in the text, since they have attracted the author's attention over the last decades and cover topics more or less related to this monograph. The interested reader could hopefully take advantage of this bibliography for enlarging and improving the scope of the monograph itself and developing new results.

This book is divided into six chapters and six appendices whose contents can be briefly summarized as follows. Since we have chosen to stress the importance of fractional calculus in modelling viscoelasticity, the first two chapters are devoted to providing an outline of the main notions in fractional calculus and linear viscoelasticity, respectively. The third chapter provides an analysis of the viscoelastic models based on constitutive equations containing integrals and derivatives of fractional order.

The remaining three chapters are devoted to wave propagation in linear viscoelastic media, so we can consider this chapter-set as a second part of the book. The fourth chapter deals with the general properties of dispersion and dissipation that characterize the wave propagation in linear viscoelastic media. In the fifth chapter we discuss asymptotic representations for viscoelastic waves generated by impact problems. In particular we deal with the techniques of wave-front expansions and saddle-point approximations. We then discuss the matching between the two above approximations carried out by the technique of rational Padè approximants. Noteworthy examples are illustrated with graphics. Finally, the sixth chapter deals with diffusion and wave-propagation problems solved with the techniques of fractional calculus. In particular, we discuss an important problem in material science: the propagation of pulses in viscoelastic solids exhibiting a constant quality factor. The tools of fractional calculus are successfully applied here because the phenomenon is shown to be governed by an evolution equation of fractional order in time.

The appendices are devoted to the special functions that play a role in the text. The most relevant formulas and plots are provided. We start in appendix A with the Eulerian functions. In appendices B, C and D we consider the Bessel, the Error and the Exponential Integral functions, respectively. Finally, in appendices E and F we analyse in detail the functions of Mittag-Leffler and Wright type, respectively. The applications of fractional calculus in diverse areas has considerably increased the importance of these functions, still ignored in most handbooks.

Francesco Mainardi
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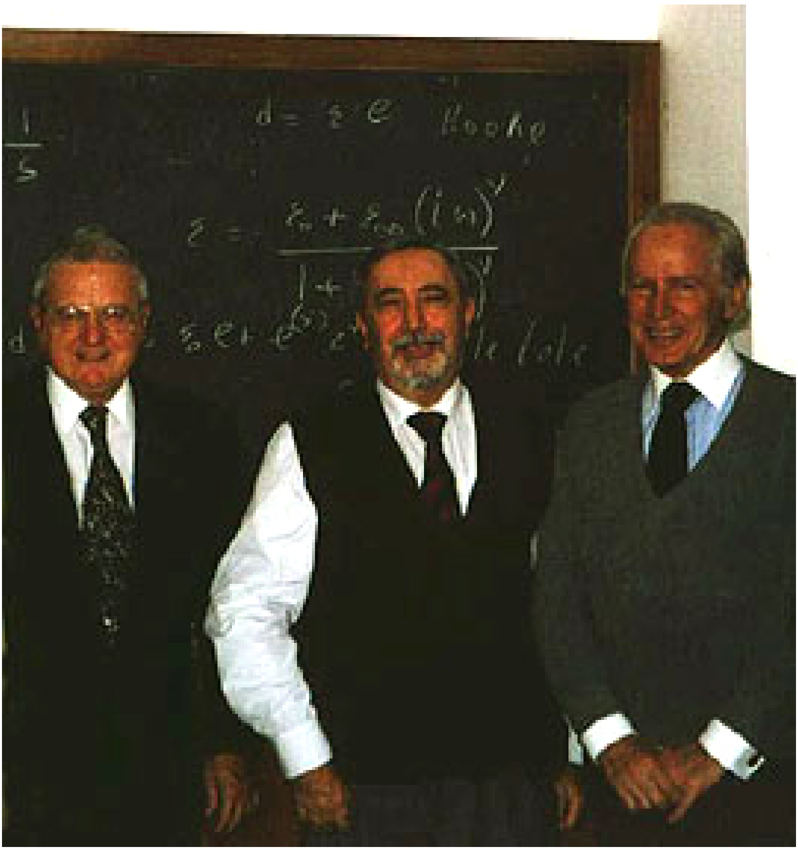

F. Mainardi between R. Gorenflo (left) and M. Caputo (right)
Bologna, April 2002

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