LECTURE NOTES ON MATHEMATICAL PHYSICS
Department of Physics, University of Bologna, Italy

THE LINEAR DIFFUSION EQUATION

Part I: Analytical Methods of Solution based on Integral Transforms

(pp. 1-18)

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1 Introduction

The simplest linear evolution equations in Mathematical Physics are the classical *diffusion* and *wave* equations. Denoting by $\mathbf{x} := \{x\,,\,y\,,\,z\}\,,\ t$ the space time variables and by $u = u(\mathbf{x}, t)$ the field variable, these equations read

$$Diffusion\ equation \qquad \boxed{\frac{\partial u}{\partial t} = D\,\nabla^2\,u}\,, \tag{1.1}$$

$$Wave\ equation \qquad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2\,u}\,, \tag{1.2}$$

where D and c are positive constants. In above equations ∇ denotes the *nabla operator* $\nabla := \left\{\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right\}$, so that $\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the *Laplacian.* Deferring the analysis of wave phenomena, now we consider the diffusion equation which is known to describe a number of physical models, such as the conduction of heat in a solid (*Fourier law*) or the spread of solute particles in a solvent (*Fick law*). The constant D is called the *diffusivity*; its dimensions are $L^2\,T^{-1}$, *i.e.* those of the kinematic viscosity of a fluid.

In these Lecture Notes we consider the one-dimensional diffusion equation,

$$\boxed{\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}}\,, \tag{1.3}$$

equipped with the most simple initial and boundary conditions, and we present the methods of solution for the related problems based on integral (Fourier - Laplace) transforms. The use of these transforms leads in a simply way to the concepts of *fundamental solutions* or *Green functions* (in the original variables x, t) and of *auxiliary function* (in one variable $z = x/\sqrt{Dt}$, referred to as the *similarity variable*).

We point out that a number of evolution equations can be reduced by appropriate transformations (change of independent and dependent variables) to the simple linear diffusion equation (1.3).

Examples of physical relevance are the (linear))*Fokker-Planck-Kolmogorov equation*

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\,\frac{\partial u}{\partial x} + u}\,, \tag{1.4}$$

found in non-equilibrium statistical mechanics and the (quasi-linear)*Burgers equation*

$$\boxed{\frac{\partial u}{\partial t} + \lambda\,u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}}\,, \quad \lambda, \nu > 0\,, \tag{1.5}$$

which is found in non linear acoustics. It must kept in mind that the last equation was just introduced by Burgers as a simple model of turbulence.

The transformation required for the *Fokker-Planck-Kolmogorov equation* (1.4) is

$$x \to \xi = x\,\mathrm{e}^{t}\,, \quad t \to \tau = \frac{\mathrm{e}^{2t} - 1}{2}\,, \quad u \to \Phi = u\,\mathrm{e}^{-t}\,, \tag{1.6}$$

which leads to

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial \xi^2}\,. \tag{1.7}$$

The transformation required for the *Burgers* equation (1.5) is the celebrated *Cole-Hopf transformation*

$$u(x,t) = -\frac{2\nu}{\lambda}\,\frac{1}{\Phi}\,\frac{\partial \Phi}{\partial x} = -\frac{2\nu}{\lambda}\,\frac{\partial}{\partial x}\ln \Phi\,, \quad \Phi = \Phi(x,t) > 0\,, \tag{1.8}$$

which leads to

$$\frac{\partial \Phi}{\partial t} = \nu\,\frac{\partial^2 \Phi}{\partial x^2}\,. \tag{1.9}$$

The *Cole-Hopf transformation* can be better understood in the following two steps

$$u = -\frac{\partial V}{\partial x} \tag{1.10}$$

$$V = \frac{2\nu}{\lambda}\ln \Phi\,. \tag{1.11}$$

The equation satisfied by $V = V(x,t)$, known as the *Potential Burgers* equation, turns out to be

$$\boxed{\frac{\partial V}{\partial t} - \frac{\lambda}{2}\left(\frac{\partial V}{\partial x}\right)^2 = \nu\,\frac{\partial^2 V}{\partial x^2}}\,. \tag{1.12}$$

2 Physical insights: heat conduction

To begin with, it is important to have a physical understanding of how (1.1) arises, and we consider the simple model of heat conduction in a solid: in this case $u(\mathbf{x},t)$ represents the temperature in a solid at position $\mathbf{x}$ and time t.

The field equation (1.1) governing the temperature is a consequence of the *Fourier law*

$$\boxed{\mathbf{q} = -K\,\nabla u}\,, \tag{2.1}$$

and the *energy balance equation* (in the absence of work, heat sources and heats sinks)

$$\boxed{\rho\,c\,\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{q} = 0}\,, \tag{2.2}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ denotes the *heat flux*, *i.e.* the amount of heat per unit area per unit time conducted across a plane cross-section, K is the *thermal conductivity*, c is the *specific heat* and ρ is the mass *density*. The quantity $\epsilon = \epsilon(\mathbf{x}, t) = \rho\, c\, u(\mathbf{x}, t)$ represents the *specific internal energy*. Eliminating $\mathbf{q}$ from (2.1-2) yields (1.1), where the *diffusivity* D turns out to be given by

$$\boxed{D = \frac{K}{\rho c}}\,. \tag{2.3}$$

In order to get more physical insight, in Table I we provide the values of the physical constants $\{\rho, c, K, D\}$ entering Eq. (2.3) for various materials (at room temperature). The units are *c.g.s.*, *calorie*, and ${}^0C$ so that in this system one measures ρ in g/cm^3, c in $cal/g\,{}^oC$, K in $cal/cm\,s\,{}^oC$ and D in cm^2/s. We point out that the diffusivity has the dimensions of a kinematic viscosity. This Table is intended only to indicate the orders of magnitude likely to occur in practice.

Substance	Density ρ	Specific heat c	Conductivity K	Diffusivity $D = K/(\rho\, c)$
Gold	19.30	0.03	0.70	1.18
Mercury	13.55	0.03	0.02	$4.4\,10^{-2}$
Silver	10.49	0.06	1.00	1.71
Copper	8.94	0.09	0.93	1.14
Brass	8.50	0.09	0.25	0.33
Zinc	7.14	0.09	0.27	0.41
Granite	2.60	0.21	$7.6\,10^{-3}$	$1.1\,10^{-2}$
Soil (average)	2.50	0.20	$2.3\,10^{-3}$	$4.6\,10^{-3}$
Glass (crown)	2.40	0.20	$2.8\,10^{-3}$	$5.8\,10^{-3}$
Concrete	2.30	0.23	$2.2\,10^{-3}$	$4.2\,10^{-3}$
Water	1.00	1.00	$1.4\,10^{-3}$	$1.4\,10^{-3}$
Air	$1.29\,10^{-3}$	0.24	$5.8\,10^{-5}$	0.19

Table I: Thermal properties of some common substances

We note from the Table that among the *metals*, the diffusivity reaches its highest value with silver, and its lowest value with mercury; among the *non-metals*, the diffusivity reaches its highest value with air, and its lowest value with water.

When the heat conduction can be considered as a one-dimensional phenomenon, *e.g.* if the solid is a thin rod extended along the x-axis, the field equations (2.1-2) reduce to

$$q(x,t) = -K\, u_x(x,t)\,, \tag{2.4}$$

$$\rho c\, \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0\,. \tag{2.5}$$

In this case the temperature $u(x,t)$ turns out to be governed by the one-dimensional diffusion equation (1.3).

3 Mathematical insights: boundary value problems

Like for any partial differential equation (*pde*) occurring in mathematical physics, we must specify some *boundary conditions* (*i.e.* the values attained by the field variable and/or by certain its derivatives on the boundary of the space-time domain) in order to guarantee the existence, the uniqueness and (hopefully) the determination of a solution of physical interest to the problem.

There are many possible *boundary conditions* for the diffusion equation (1.3). To avoid unproductive generalities, we consider the most simple boundary conditions, the physical meaning of which suggests that they are sufficient to determine the solution $u(x,t)$. In this respect, for easier visualization, we could retain the terminology of heat transfer.

As far as the space-time domain is concerning, we presume that t varies in the semi-infinite interval $0 \le t < +\infty$, while the variable x may range in an interval which may be bounded or unbounded at one or both sides. Accordingly, we have in the $x - t$ plain, as fundamental region of the *pde* (1.3), either a half-strip or a quadrant or a half-plane.

As far as the boundary values are concerning, retaining the terminology of heat transfer, we specify the *initial temperature* of the conductor by a function of x, say $f(x)$, and the *temperature at the end points* by two functions of t, say $g(t)$, $h(t)$. More precisely, these *boundary conditions* are understood as limits as $(x\,,\,t)$ approaches the respective boundary along a line orthogonal to it; in mathematical terms, with $-\infty \le a < b \le +\infty$, we write the boundary conditions as follows

$$\boxed{\lim_{t\to 0^+} u(x,t) := u(x,0^+) = f(x)\,, \quad a < x < b}\,, \tag{3.1}$$

and

$$\boxed{\begin{cases} \displaystyle\lim_{x\to a^+} u(x,t) := u(a^+,t) = g(t)\,, \\ \displaystyle\lim_{x\to b^-} u(x,t) := u(b^-,t) = h(t)\,, \end{cases} \quad t > 0}\,. \tag{3.2}$$

For practical purposes, if the medium is bounded at both sides or unbounded at one side, it may be convenient to refer to the intervals $0 < x < L$ or $0 < x < +\infty$, respectively.

Because of the linearity of the diffusion equation, the above boundary value problem (BVP) [(1.3)+ (3.1-2)] can be formatted as the superposition of three distinct BVP as follows.

Denoting by $\mathcal{D}$ the differential operator of diffusion, *i.e.*

$$\mathcal{D} := \frac{\partial}{\partial t} - D\,\frac{\partial^2}{\partial x^2}\,, \tag{3.3}$$

and writing

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)\,, \tag{3.4}$$

we require

$$(a)\quad \mathcal{D}\,u_1(x,t) = 0\,,\quad u_1(x,0^+) = f(x)\,,\; u_1(a^+,t) = 0\,,\quad u_1(b^-,t) = 0\,; \tag{3.5a}$$

$$(b)\quad \mathcal{D}\,u_2(x,t) = 0\,,\quad u_2(x,0^+) = 0\,,\quad u_2(a^+,t) = g(t)\,,\; u_2(b^-,t) = 0\,; \tag{3.5b}$$

$$(c)\quad \mathcal{D}\,u_3(x,t) = 0\,,\quad u_3(x,0^+) = 0\,,\quad u_3(a^+,t) = 0\,,\quad u_3(b^-,t) = h(t)\,. \tag{3.5c}$$

The given functions f , g , h , usually referred to as *data functions* are requested to satisfy some regularity conditions. Here we intend to privilege the application of transform methods based on the space FOURIER transform and the time LAPLACE transform to find the solution in the space-time domain. Therefore, for our purposes, the following requirements for the *data functions* are sufficient: the space function $f(x)$ must admit the Fourier transform (if their support is finite, the Fourier series expansion), whereas the time functions $g(t)$, $h(t)$ must admit the Laplace transform.

Having in mind the application of the Laplace transform in the time variable, we have implicitly assumed, for $t < 0$, the medium to be quiescent (at a constant equilibrium temperature, using the terminology of heat transfer); without loosing in generality, we require $u(x,t) \equiv u(x,0^-) \equiv 0$ for $a < x < b\,,\; t < 0$. This implies that in our approach any function of t is assumed to be *causal.* For a *causal function* we mean a function $\phi(t)$ vanishing for $t < 0$ so that, sometimes, it may be convenient to point out this fact writing

$$\phi_+(t) := \phi(t)\,\Theta(t)\,,\quad \text{where}\quad \Theta(t) := \begin{cases} 1 & \text{if} \quad t > 0\,, \\ 0 & \text{if} \quad t < 0\,, \end{cases} \tag{3.6}$$

denotes the *Heaviside* or *unit-step* function. For this function it is usual to find alternative notations, as $H(t)$ or $1(t)$.

4 Cauchy and Signalling problems: the results

The two basic problems for the diffusion equation are usually referred to as the *Cauchy problem* and the *Signalling problem*. The former is provided by (2.10a) with $a = -\infty\,,\ b = +\infty\,$, the latter by (2.10b) with $a = 0\,,\ b = +\infty\,$. In this Section we would like to present in detail the results concerning the above problems.

In the *Cauchy problem* the medium, supposed unlimited ($-\infty < x < +\infty$), is subjected, for $t = 0$, to a known disturbance, provided by a function $f(x)$; the conditions thus read

$$\begin{cases} u(x,0^+) = f(x)\,, & -\infty < x < +\infty\,; \\ u(\pm\infty,t) = 0\,, & t > 0\,. \end{cases} \tag{4.1}$$

Since the boundary values which are specified along the boundary $t = 0$, usually are called *initial values*, this problem can be considered a pure *initial-value problem* (IVP).

In the *Signalling problem* the medium, supposed semi-infinite ($0 \le x < +\infty$) and initially undisturbed, is subjected, at $x = 0$ (the accessible end) and for $t > 0$, to a known disturbance, provided by a causal function $g(t)$; the conditions thus read

$$\begin{cases} u(x,0^+) = 0\,, & 0 < x < +\infty\,; \\ u(0^+,t) = g(t)\,,\ u(+\infty,t) = 0\,, & t > 0\,. \end{cases} \tag{4.2}$$

This problem is also referred to as the *initial boundary value problem* ($IBVP$) in the quadrant $\{x\,,\,t\} > 0\,$.

The next Sections are devoted to find the solutions to the above problems by using methods based on integral transforms. Hereafter we resume the results. For each problem the solution turns out to be expressed by a proper convolution between the source function and a characteristic function (the so-called *Green function* or *fundamental solution* of the problem) according to the following scheme.

$$\textit{Cauchy problem}\ :\ \boxed{u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t)\, f(x-\xi)\, d\xi = \mathcal{G}_c(x,t) * f(x)}\,, \tag{4.3}$$

where $*$ denotes the (bilateral) space convolution. The function $\mathcal{G}_c(x,t)$, referred to as the *Green function* for the *Cauchy problem*, turns out to be

$$\boxed{\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi D}}\, t^{-1/2}\, \mathrm{e}^{-x^2/(4D\,t)}}\,. \tag{4.4}$$

It represents the solution for $f(x) = \delta(x)$, where $\delta(x)$ denotes the *Dirac* or *delta* generalized function.

$$\textit{Signalling problem} \ : \quad \boxed{u(x,t) = \int_0^t \mathcal{G}_s(x,\tau)\, g(t-\tau)\, d\tau = \mathcal{G}_s(x,t) \, * \, g(t)}\,, \tag{4.5}$$

where now $*$ denotes the (unilateral) time convolution. (For causal functions the unlimited interval of the integral simply reduces to $[0\,,\,t]$). The function $\mathcal{G}_s(x,t)$, referred to as the *Green function* for the *Signalling problem*, turns out to be

$$\boxed{\mathcal{G}_s(x,t) = \frac{x}{2\sqrt{\pi D}}\, t^{-3/2}\, \mathrm{e}^{-x^2/(4D\,t)}}\,. \tag{4.6}$$

It represents the solution of the problem for $g(t) = \delta_+(t)$, where $\delta_+(t)$ is denoting the *causal delta generalized function.*

We point out that, because of their relevance to construct the appropriate solution for any Cauchy or Signalling problem, the Green functions (4.4) and (4.6) are also referred to as the *fundamental solutions* for the respective problems. At the end of these Lecture Notes some plots of $\mathcal{G}_c(x,t)$, $\mathcal{G}_s(x,t)$ are presented, either versus x at fixed t, or versus t at fixed x, [see Figs. 1-4].

Introducing in Eqs (4.3)-(4.4)the new variable of integration $\eta = \eta(\xi)$,

$$\eta = \frac{\xi}{2\sqrt{D\,t}}\,, \tag{4.7}$$

we obtain an alternative representation for the general solution of the *Cauchy problem*,

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta^2}\, f\left(x - 2\,\sqrt{D\,t}\,\eta\right)\, d\eta\,. \tag{4.8}$$

Introducing in Eqs (4.5)-(4.6) the new variable of integration $\sigma = \sigma(\tau)$,

$$\sigma = \frac{x}{2\sqrt{D\,\tau}}\,, \tag{4.9}$$

we obtain an alternative representation of the general solution of the *Signalling problem*,

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{D\,t}}^{\infty} \mathrm{e}^{-\sigma^2}\, g\left(t - \frac{x^2}{4D\,\sigma^2}\right)\, d\sigma\,. \tag{4.10}$$

A noteworthy particular case of (4.10) is obtained when $g(t) = \Theta(t)$. In this case the solution turns out to be

$$\boxed{u(x,t) := \mathcal{H}_s(x,t) = \mathrm{erfc}\, \left(\frac{x}{2\sqrt{D\,t}}\right)}\,, \tag{4.11}$$

where erfc denotes the *complementary error function*, see Remark at the end of this Section.

Plots of the function $\mathcal{H}_s$, which is referred to as the *step response* for the Signalling problem, are presented, either versus x at fixed t, or versus t at fixed x, in Figs 5-6, respectively, at the end of the Lecture Notes. This solution is related to the corresponding *Green function* by the relation

$$\boxed{\mathcal{H}_s(x,t) = \int_0^t \mathcal{G}_s(x,\tau)\, d\tau} \,. \tag{4.12}$$

In the next Sections 5 and 6 the functions $\mathcal{G}_c$ and $\mathcal{G}_s$ will be derived by using the techniques of Fourier and Laplace transforms, respectively.

We note from Eqs (4.4), (4.6) that the following relevant property is valid for $\{x\,,t\} > 0\,,$

$$\boxed{x\,\mathcal{G}_c(x,t) = t\,\mathcal{G}_s(x,t) = F(z)}\,, \tag{4.13}$$

where

$$\boxed{z = \frac{x}{\sqrt{D\,t}}\,, \quad F(z) = \frac{z}{2}\,M(z)\,, \quad M(z) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-z^2/4}}\,. \tag{4.14}$$

The first equality in (4.13) can be referred to as the *reciprocity relation* between the two Green functions. In Eqs (4.13)-(4.14) z represents the *similarity variable* and $F(z)\,,\ M(z)$ are referred to as the *auxiliary functions* in that through them the Green functions can easily be derived. The analyticity properties of the auxiliary function $M(z)$, as series and integral representations in the complex plane, will be discussed later, in the Lecture Notes devoted to the *time fractional diffusion-wave equation.*

Remark: We recall that the *error function* $\mathrm{erf}\,(x)$ is (usually) defined as

$$\boxed{\mathrm{erf}\,(x) := \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-\xi^2}\, d\xi\,, \quad x \in \mathbb{R}}\,. \tag{4.15}$$

We note: $\mathrm{erf}\,(-x) = -\mathrm{erf}\,(x)$ with $\mathrm{erf}\,(\pm\infty) = \pm 1$. Its Taylor series (around $x = 0$) reads

$$\boxed{\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}\, x^{2n+1} = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-x^2} \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!}\, x^{2n+1}}\,. \tag{4.16}$$

Its derivative is $\mathrm{erf}\,'(x) = 2\,\mathrm{e}^{-x^2}/\sqrt{\pi}$. The *complementary error function* $\mathrm{erfc}\,(x)$ is

$$\boxed{\mathrm{erfc}\,(x) := 1 - \mathrm{erf}\,(x) := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \mathrm{e}^{-\xi^2}\, d\xi\,, \quad x \in \mathbb{R}}\,. \tag{4.17}$$

We point out the asymptotic representation: $\mathrm{erfc}\,(x) \sim \mathrm{e}^{-x^2}/(\sqrt{\pi}\,x)$, as $x \to \infty$, that provides a good approximation already from $x \geq 1.5$. To get more insight on the family of the error functions $\{\mathrm{erf}\,(x),\ \mathrm{erf}\,'(x),\ \mathrm{erfc}\,(x)\}$ see Fig. 7 and Table II.

5 The Green function for the Cauchy problem via Fourier transform

For the *Cauchy Problem* the use of the Fourier transform (FT) with respect to x is straightforward. We adopt the following notation ($\kappa \in \mathbb{R}$)

$$\boxed{\widehat{u}(\kappa,t) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mp i\kappa x} u(x,t)\,dx \;\div\; u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\pm i\kappa x}\widehat{u}(\kappa,t)\,d\kappa}\,.$$

Applying FT to [(1.3)+(4.1)] we get the first-order differential equation in t,

$$\frac{d\widehat{u}}{dt} + D\kappa^2\,\widehat{u} = 0\,, \quad t>0\,, \tag{5.1}$$

with the initial condition

$$\widehat{u}(\kappa,0^+) = \widehat{f}(\kappa)\,. \tag{5.2}$$

The solution of this initial value problem is

$$\widehat{u}(\kappa,t) = \widehat{f}(\kappa)\,\mathrm{e}^{-Dt\,\kappa^2}\,. \tag{5.3}$$

Introducing

$$\boxed{\widehat{\mathcal{G}_c}(\kappa,t) := \mathrm{e}^{-D\,t\,\kappa^2}}\,, \tag{5.4}$$

the transform solution (5.3) reads

$$\widehat{u}(\kappa,t) = \widehat{f}(\kappa)\,\widehat{\mathcal{G}_c}(k,t)\,. \tag{5.5}$$

Then, applying the convolution theorem for Fourier transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t)\,f(x-\xi)\,d\xi = \mathcal{G}_c(x,t) * f(x)}\,, \tag{5.6}$$

where $\mathcal{G}_c(x,t)$ is the *Green function* for the *Cauchy problem.* For the inversion of $\widehat{\mathcal{G}_c}(\kappa,t)$ we recall the Fourier transform pair [see *e.g.* Ghizzetti-Ossicini, pp. 69-70; Papoulis, pp. 24-25]

$$\boxed{\mathrm{e}^{-a\,\kappa^2} \;\div\; \frac{1}{2\sqrt{\pi\,a}}\,\mathrm{e}^{-x^2/(4a)} \quad (a>0)}\,, \tag{5.7}$$

which yields to ($a = Dt$)

$$\boxed{\mathcal{G}_c(x,t) = \frac{1}{2\sqrt{\pi\,D\,t}}\,\mathrm{e}^{-x^2/(4D\,t)}}\,. \tag{5.8}$$

6 The Green function for the Signalling problem via Laplace transform

The *Signalling Problem* is conveniently treated by the Laplace transform (LT) technique with respect to t. We adopt the following notation ($s \in \mathbb{C}$)

$$\boxed{\widetilde{u}(x,s) := \int_0^{+\infty} \mathrm{e}^{-st} u(x,t)\, dt \;\div\; u(x,t) = \frac{1}{2\pi i} \int_{Br} \mathrm{e}^{st} \widetilde{u}(x,s)\, ds} .$$

Applying LT to [(1.3)+(4.2)] we get the second-order differential equation in x,

$$\frac{d^2\widetilde{u}}{dx^2}(x,s) - \frac{s}{D}\, \widetilde{u}(x,s) = 0\,, \quad x > 0\,, \tag{6.1}$$

with the boundary conditions

$$\widetilde{u}(0^+, s) = \widetilde{g}(s)\,, \quad \widetilde{u}(+\infty, s) = 0\,. \tag{6.2}$$

Solving (6.1) we find $\widetilde{u}(x,s) = c_1(s)\, \mathrm{e}^{-(x/\sqrt{D})\, s^{1/2}} + c_2(s)\, \mathrm{e}^{+(x/\sqrt{D})\, s^{1/2}}$; then, choosing $c_1(s) = \widetilde{f}(s)$ and $c_2(s) = 0$ to satisfy the boundary conditions (6.2), we get

$$\widetilde{u}(x,s) = \widetilde{g}(s)\, \mathrm{e}^{-(x/\sqrt{D})\, s^{1/2}} . \tag{6.3}$$

Introducing

$$\boxed{\widetilde{\mathcal{G}}_s(x,s) := \mathrm{e}^{-(x/\sqrt{D})\, s^{1/2}}}\,, \tag{6.4}$$

the transform solution (6.3) reads

$$\widetilde{u}(x,s) = \widetilde{g}(s)\, \widetilde{\mathcal{G}}_s(x,s)\,. \tag{6.5}$$

Then, applying the convolution theorem for Laplace transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \int_0^t \mathcal{G}_s(x,\tau)\, g(t-\tau)\, d\tau = \mathcal{G}_s(x,t) \,*\, g(t)}\,, \tag{6.6}$$

where $\mathcal{G}_s(x,t)$ is the *Green function* for the *Signalling problem.* For the inversion of $\widetilde{\mathcal{G}}_s(x,s)$ we recall the Laplace transform pair

$$\boxed{\mathrm{e}^{-a\, s^{1/2}} \;\div\; \frac{a}{2\sqrt{\pi}}\, t^{-3/2}\, \mathrm{e}^{-a^2/(4t)} := \psi(a,t) \quad (a > 0)}\,, \tag{6.7}$$

which yields ($a = x/\sqrt{D}$)

$$\boxed{\mathcal{G}_s(x,t) = \frac{x}{2\sqrt{\pi\, D\, t^3}}\, \mathrm{e}^{-x^2/(4D\, t)}} . \tag{6.8}$$

Let us now consider the noteworthy particular case $g(t) = \Theta(t)$ that provides the *step response* $\mathcal{H}_s(x,t) = \int_0^t \mathcal{G}_s(x,\tau)\, d\tau$ according to Eq. (4.12). In fact, from Eqs (6.4)-(6.5), we have:

$$\widetilde{\mathcal{H}_s}(x,s) = \frac{\widetilde{\mathcal{G}_s}(x,s)}{s} \,. \tag{6.9}$$

For the inversion of $\widetilde{\mathcal{H}_s}(x,s)$ we recall the transform pair

$$\boxed{\frac{\mathrm{e}^{-a\, s^{1/2}}}{s} \div \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) := \phi(a,t) \quad (a > 0)}\,, \tag{6.10}$$

which yields ($a = x/\sqrt{D}$)

$$\boxed{\mathcal{H}_s(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{D\,t}}\right)}\,. \tag{6.11}$$

in agreement with (4.11).

An interesting signalling problem is concerning the medium, supposed semi-infinite ($0 \le x < +\infty$) and initially undisturbed, when it is subjected, at $x = 0$ (the accessible end) and for $t > 0$, to a known gradient (*heat flux* for the heat conduction model) provided by a causal function

$$q(t) = -K\, \frac{\partial u}{\partial x}\bigg|_{x=0} \,. \tag{6.12}$$

We refer to it as to the *flux-signalling problem.* In this case the conditions read

$$\begin{cases} u(x,0^+) = 0\,, & 0 < x < +\infty\,; \\ \dfrac{\partial u}{\partial x}(0^+,t) = -q(t)/K\,,\ u(+\infty,t) = 0\,, & t > 0\,. \end{cases} \tag{6.13}$$

Applying *LT* to [(1.3)+(6.13)] we get the (usual) second-order differential equation in x,

$$\frac{d^2\widetilde{u}}{dx^2}(x,s) - \frac{s}{D}\,\widetilde{u}(x,s) = 0\,, \quad x > 0\,, \tag{6.1}$$

with the (new) boundary conditions

$$\frac{d\widetilde{u}}{dx}(0^+,s) = -\widetilde{q}(s)/K\,, \quad \widetilde{u}(+\infty,s) = 0\,. \tag{6.14}$$

Note the difference between [(6.1)+(6.2)] and [(6.1)+(6.14)]. Solving (6.1) we find

$$\widetilde{u}(x,s) = c_1(s)\,\mathrm{e}^{-(x/\sqrt{D})\,s^{1/2}} + c_2(s)\,\mathrm{e}^{+(x/\sqrt{D})\,s^{1/2}}\,;$$

and hence, choosing

$$c_1(s) = \frac{\sqrt{D}}{K} \frac{\widetilde{q}(s)}{s^{1/2}}, \quad c_2(s) = 0,$$

to satisfy the boundary conditions (6.14), we get

$$\widetilde{u}(x,s) = \frac{\sqrt{D}}{K} \widetilde{q}(s) \frac{\mathrm{e}^{-(x/\sqrt{D})\, s^{1/2}}}{s^{1/2}}. \tag{6.15}$$

Introducing

$$\boxed{\widetilde{\mathcal{G}}_q(x,s) := \frac{\mathrm{e}^{-(x/\sqrt{D})\, s^{1/2}}}{s^{1/2}}}, \tag{6.16}$$

the transform solution (6.15) reads

$$\widetilde{u}(x,s) = \frac{\sqrt{D}}{K} \widetilde{q}(s)\, \widetilde{\mathcal{G}}_q(x,s). \tag{6.17}$$

Then, applying the convolution theorem for Laplace transforms, the solution in the space-time domain turns out to be

$$\boxed{u(x,t) = \frac{\sqrt{D}}{K} \int_0^t \mathcal{G}_q(x,\tau)\, q(t-\tau)\, d\tau = \frac{\sqrt{D}}{K} \mathcal{G}_q(x,t) \, * \, q(t)}, \tag{6.18}$$

where $\mathcal{G}_q(x,t)$ is the *Green function* for the *Heat-Flow Signalling problem.* For the inversion of $\widetilde{\mathcal{G}}_q(x,s)$ we recall the Laplace transform pair

$$\boxed{\frac{\mathrm{e}^{-a\, s^{1/2}}}{s^{1/2}} \div \frac{1}{\sqrt{\pi}} t^{-1/2}\, \mathrm{e}^{-a^2/(4t)} := \chi(a,t) \quad (a > 0)}, \tag{6.19}$$

which yields ($a = x/\sqrt{D}$)

$$\boxed{\mathcal{G}_q(x,t) = \frac{1}{\sqrt{\pi\, D\, t}}\, \mathrm{e}^{-x^2/(4D\, t)}}. \tag{6.20}$$

Plots of the Green function $\mathcal{G}_q$ are presented, either versus x at fixed t, or versus t at fixed x, in Figs 8-9, respectively, at the end of the Lecture Notes. We note that $\mathcal{G}_q = 2\mathcal{G}_c$, compare (6.20) with (5.8) for $x \geq 0$.

We point out the relevance in diffusion problems of the three functions $\phi(a,t)$, $\psi(a,t)$ and $\chi(a,t)$. We note that the relations among the three functions turn out to be easily derived by working in the Laplace transform domain. In view of these relations we like to call these functions the *three sisters*!

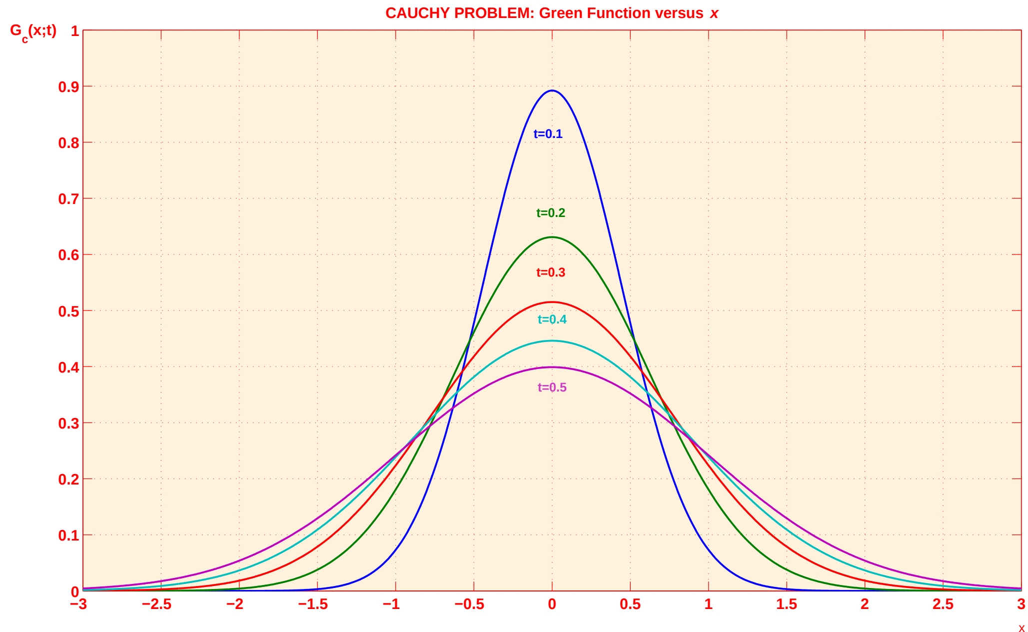

Fig 1: The Green function for the Cauchy problem versus x.

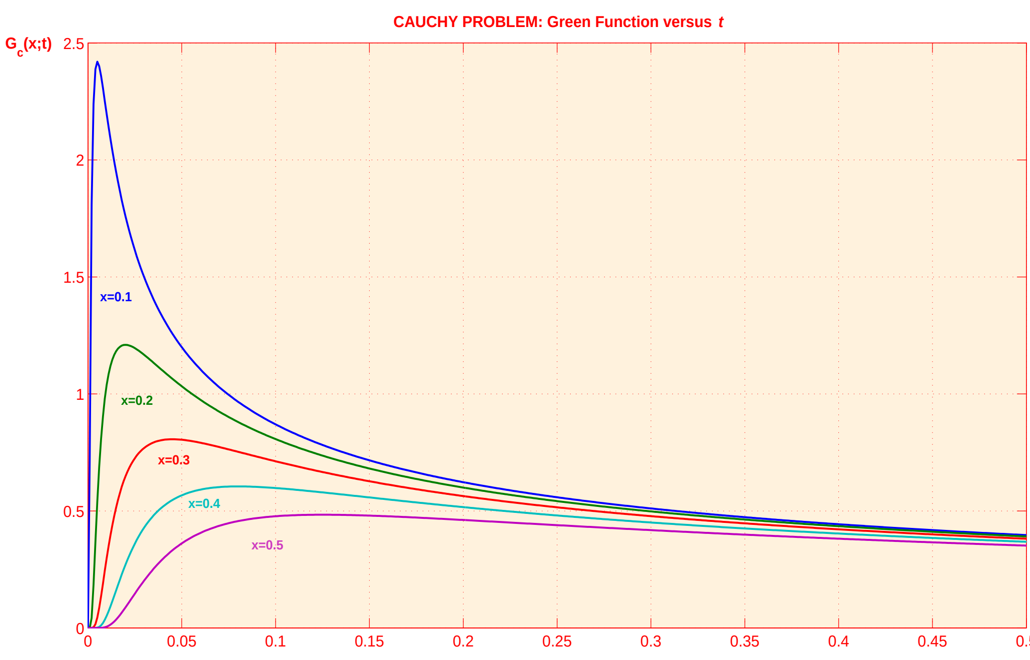

Fig 2: The Green function for the Cauchy problem versus t.

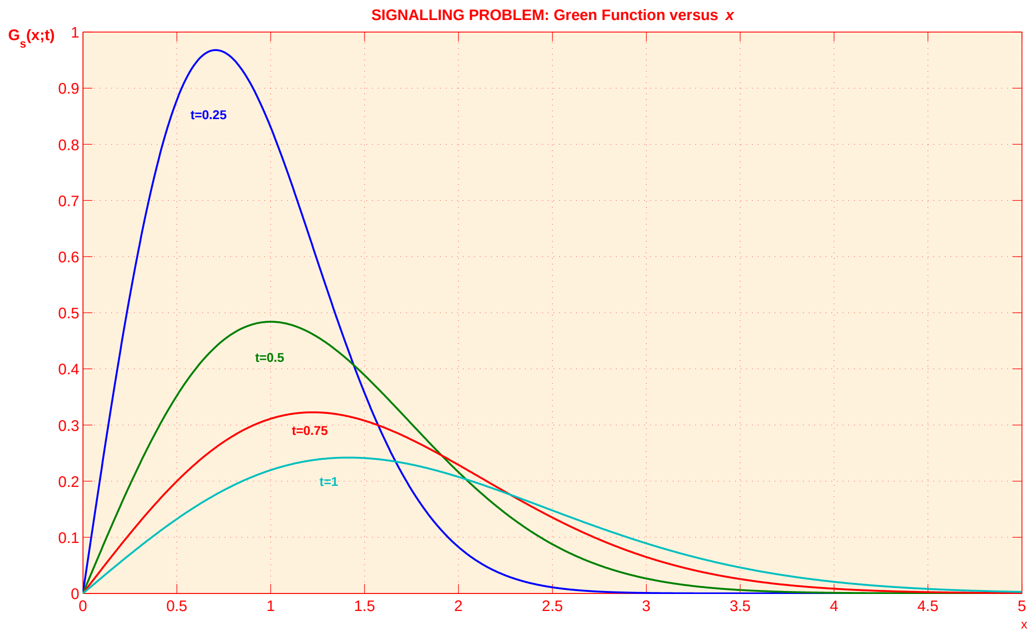

Fig 3: The Green function for the Signalling problem versus x.

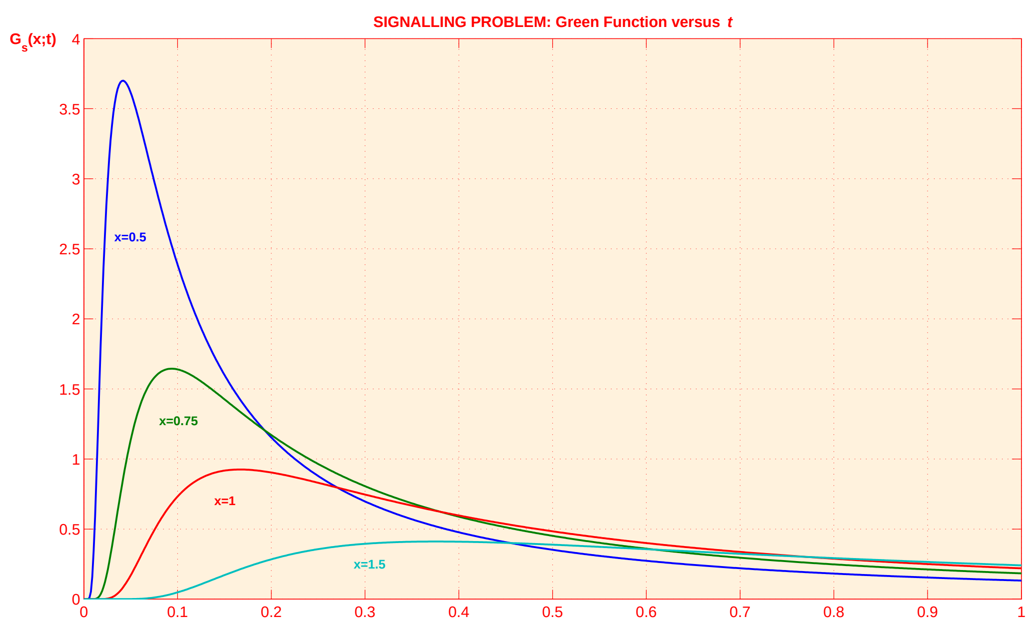

Fig 4: The Green function for the Signalling problem versus t.

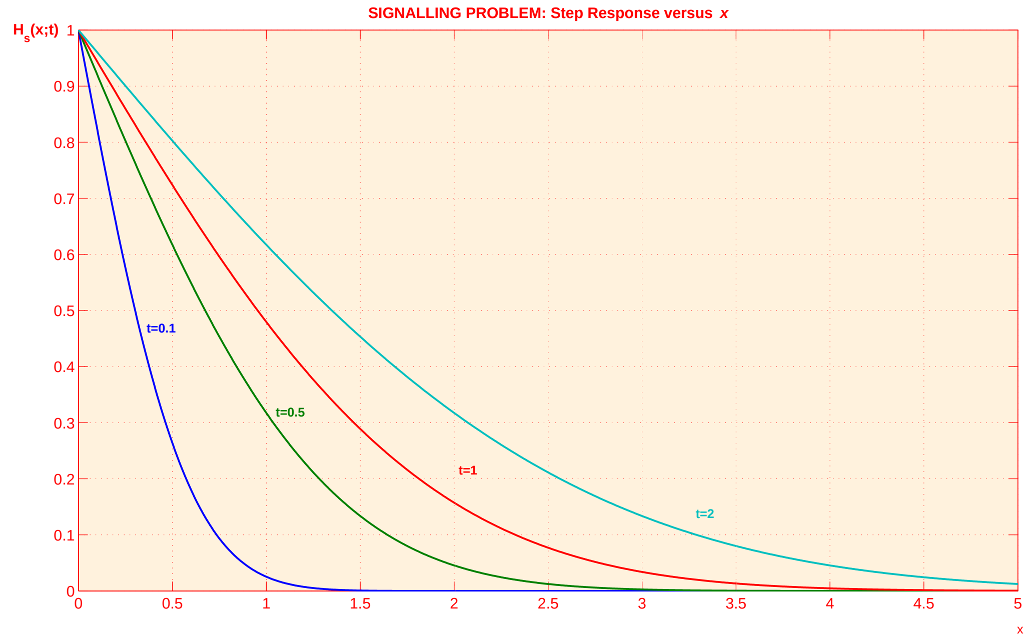

Fig 5: The step response for the Signalling problem versus x.

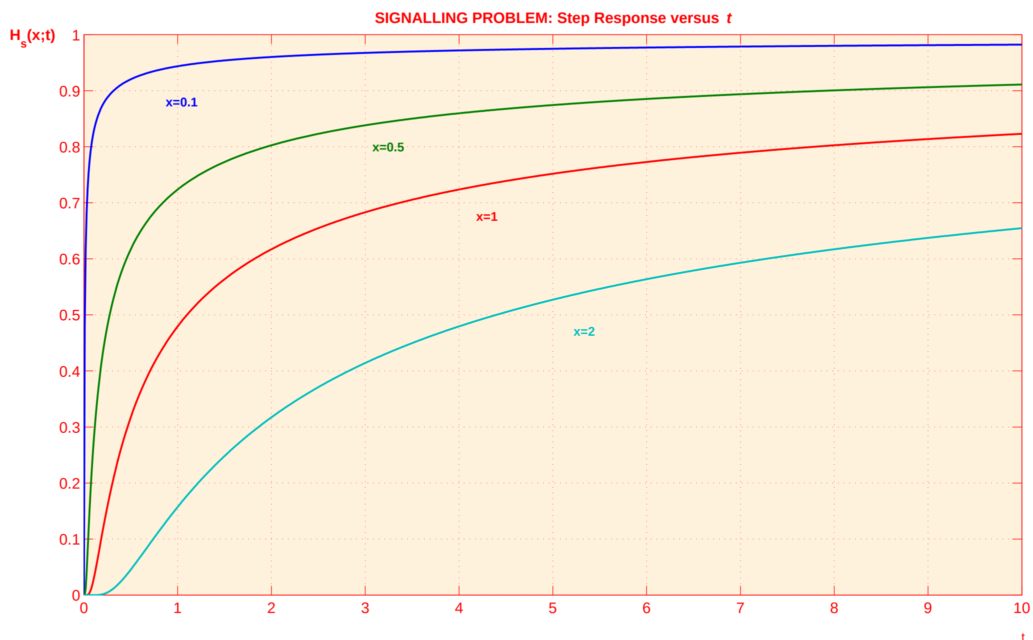

Fig 6: The step response for the Signalling problem versus t.

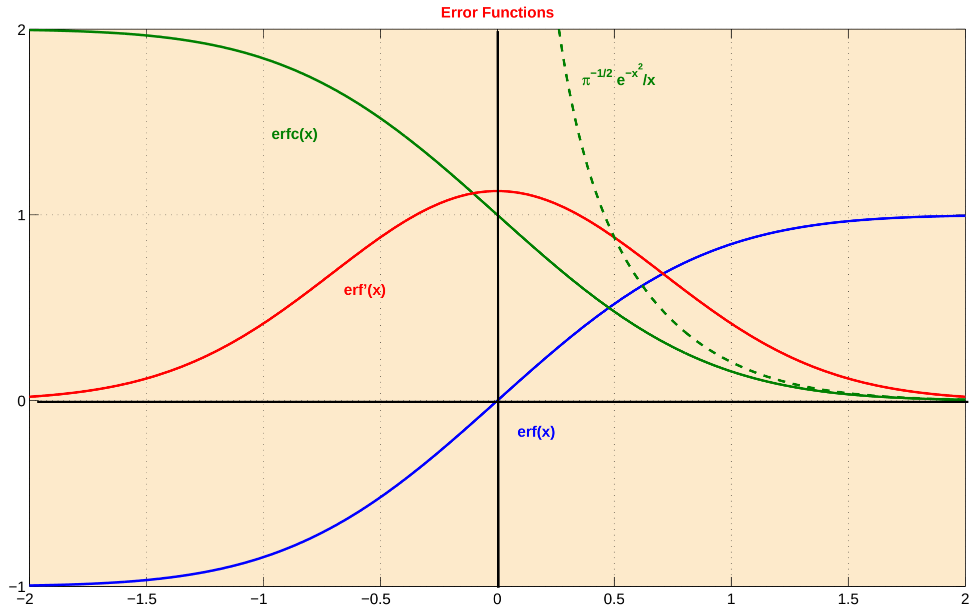

Fig 7: Plots of erf (x), erf$'(x)$ and erfc(x) in the interval $-2 \leq x \leq +2$.

x	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.50	2.00
erf (x)	0.00	0.12	0.22	0.33	0.43	0.52	0.60	0.68	0.74	0.80	0.84	0.97	0.99
erf$'(x)$	1.13	1.12	1.08	1.03	0.96	0.88	0.79	0.69	0.59	0.50	0.42	0.12	0.02
erfc(x)	1.00	0.88	0.78	0.67	0.57	0.48	0.40	0.32	0.26	0.20	0.16	0.03	0.01

Table II: Selected values of erf (x), erf$'(x)$ and erfc(x) in the interval $0 \leq x \leq 2$.

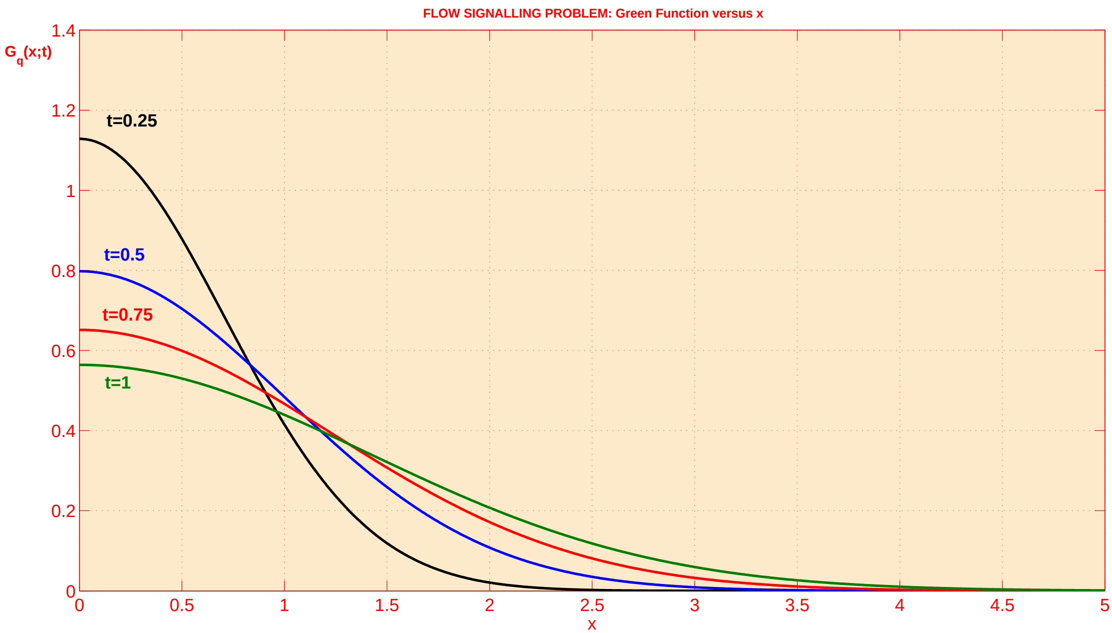

Fig 8: The Green function for the flux-signalling problem versus x.

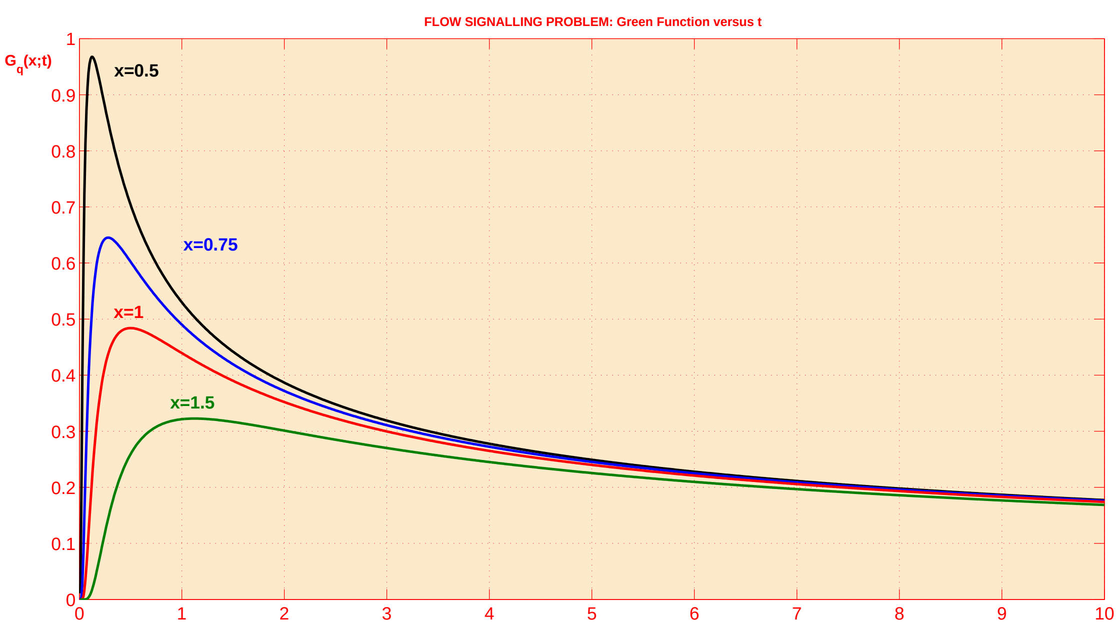

Fig 9: The Green function for the flux-signalling problem versus t.