The Two Forms of Fractional Relaxation of Distributed Order

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Abstract: The first-order differential equation of exponential relaxation can be generalized by using either the fractional derivative in the Riemann–Liouville (R-L) sense and in the Caputo (C) sense, both of a single order less than 1. The two forms turn out to be equivalent. When, however, we use fractional derivatives of distributed order (between zero and 1), the equivalence is lost, in particular on the asymptotic behaviour of the fundamental solution at small and large times. We give an outline of the theory providing the general form of the solution in terms of an integral of Laplace type over a positive measure depending on the order-distribution. We consider in some detail two cases of fractional relaxation of distribution order: the double-order and the uniformly distributed order discussing the differences between the R-L and C approaches. For all the cases considered we give plots of the solutions for moderate and large times.

Key words: Fractional relaxation, fractional calculus, Mittag–Leffler function, complete monotonicity, slowly varying functions

1. INTRODUCTION

The purpose of this paper is to study two types of fractional generalization of the classical relaxation equation. One type uses the fractional derivative in the sense of Riemann and Liouville, the other in the sense of Caputo. In its uses we distinguish between single and distributed orders of fractional derivatives.

The plan of the paper is as follows. In Section 2, we recall the relevant properties of the fractional relaxation equations of a single order $\beta = \beta_1 \in (0, 1]$, in which the fractional derivative is intended in the Riemann–Liouville (R-L) sense and in the Caputo (C) sense. The two forms are shown to be equivalent and the common solutions corresponding to a

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Figures 1–4 appear in color online: http://jvc.sagepub.com

few orders are plotted. In Section 3 we consider two general cases of fractional relaxation where in the R-L setting the order $1-\beta$, in the C-setting the order β is distributed according to a non-negative weight function $p(\beta)$. For the fundamental solutions of these equations we provide a general formula obtained by the Titchmarsh theorem on Laplace inversion. By virtue of this, these solutions appear as real Laplace transforms of a positive spectral function, and hence they are completely monotonic functions for $t \geq 0$ in analogy with the fundamental solution of the fractional relaxation equation of single order. In Section 4 we consider two typical cases of weight function: the case of two distinct orders $0 < \beta_1 < \beta_2 \leq 1$ and the case of uniform distribution of orders between zero and 1. For these cases, by using Tauberian theory, we provide asymptotic expressions of the fundamental solution near zero and near infinity, that show the different role played by the order-distribution in the R-L and C approaches. Finally, concluding remarks are given in Section 5. For the reader's convenience we briefly recall in an Appendix the essentials of Fractional Calculus useful for understanding the notions of fractional derivative in the R-L sense and in the C sense.

2. FRACTIONAL RELAXATION OF SINGLE ORDER

The classical phenomenon of relaxation in its simplest form is known to be governed by a linear ordinary differential equation of order one, possibly non-homogeneous, that hereafter we recall with the corresponding solution. Denoting by $t \geq 0$ the time variable, $u = u(t)$ the field variable, and by ${}_tD^1$ the first-order time derivative, the *relaxation* differential equation (of homogeneous type) reads

$$ {}_tD^1 u(t) = -\lambda u(t), \quad t \geq 0, \tag{2.1}$$

where λ is a positive constant denoting the inverse of some characteristic time. The solution of (2.1), under the initial condition $u(0^+) = 1$, is called the *fundamental solution* and reads

$$ u(t) = \mathrm{e}^{-\lambda t}, \quad t \geq 0. \tag{2.2}$$

From the view-point of Fractional Calculus (for a short review, see the Appendix) there appear in the literature two ways of generalizing the equation (2.1), one using the R-L, the other using the C fractional derivative. Adopting the notation of the Appendix for the two derivatives (see (A.5) and (A.6)), and denoting by β_1 the common fractional order, the two forms read, respectively, $t \geq 0$

$$ {}_tD^1 u(t) = -\lambda \, {}_tD^{1-\beta} u(t), \quad 0 < \beta \leq 1, \tag{2.3}$$

and

$$ {}_tD_*^{\beta} u(t) = -\lambda u(t), \quad 0 < \beta \leq 1, \tag{2.4}$$

where now the positive constant λ has dimensions $[t]^{-\beta}$. If we assume the same initial condition, e.g. $u(0^+) = 1$, it is not difficult to show the equivalence of the two forms by playing with the operators of standard and fractional integration and differentiation.[2]

By applying in equations (2.3) and (2.4) the technique of the Laplace transforms for fractional derivatives of C and R-L type (see (A.13)-(A.15)), we get the same result for the fundamental solution, namely

$$\widetilde{u}(s) = \frac{s^{\beta-1}}{s^{\beta}+\lambda}, \tag{2.5}$$

that, with the Mittag–Leffler function[3] E_β, yields in the time domain

$$u(t) = E_\beta(-\lambda t^\beta), \quad 0 < \beta \leq 1. \tag{2.6}$$

We agree to refer to the equation (2.3) or (2.4) as the *simple fractional relaxation equation* in the R-L or C sense, respectively.

In Figure 1 we show the solution (2.6) for a few values of the order $\beta = \beta_1$, $\beta_1 = 1/4, 1/2, 3/4, 1$, by assuming $\lambda = 1$: in the top diagram for the time interval [0, 10] (linear scales), and in the bottom diagram for the time interval $[10^1, 10^7]$ (logarithmic scales). In the lower diagram we have added in dotted lines the asymptotic values for $t \to \infty$ in order to better visualize the power-law decay expressed by $t^{-\beta_1}/\Gamma(1-\beta_1)$ for the cases $0 < \beta_1 < 1$, whereas the case $\beta_1 = 1$ is not visible in view of the faster exponential decay. In both diagrams we have shown in a dashed line the singular solution for the limiting case $\beta_1 = 0$, stretching the definition of the Mittag–Leffler function to $E_0(z) = 1/(1-z)$, the geometric series,

$$u(t) = \begin{cases} E_0(0) = 1, & t = 0, \\ E_0(-t^0) \equiv E_0(-1) = 1/2, & t > 0, \end{cases} \tag{2.7}$$

3. FRACTIONAL RELAXATION OF DISTRIBUTED ORDER

3.1. The two forms of fractional relaxation

The simple fractional relaxation equations (2.3) and (2.4) can be generalized by using the notion *fractional derivative of distributed order*.[4] We thus consider the so-called *distributed order fractional relaxation equation* or *fractional relaxation equation of distributed order*, in the two alternative forms involving the R-L and the C derivatives, that we write respectively as

$$ {}_tD^1 u(t) = -\lambda \int_0^1 p(\beta)\, {}_tD^{1-\beta} u(t)\, d\beta, \tag{3.1}$$

and

$$\int_0^1 p(\beta)\, {}_tD_*^{\beta} u_*(t)\, d\beta = -\lambda u_*(t), \tag{3.2}$$

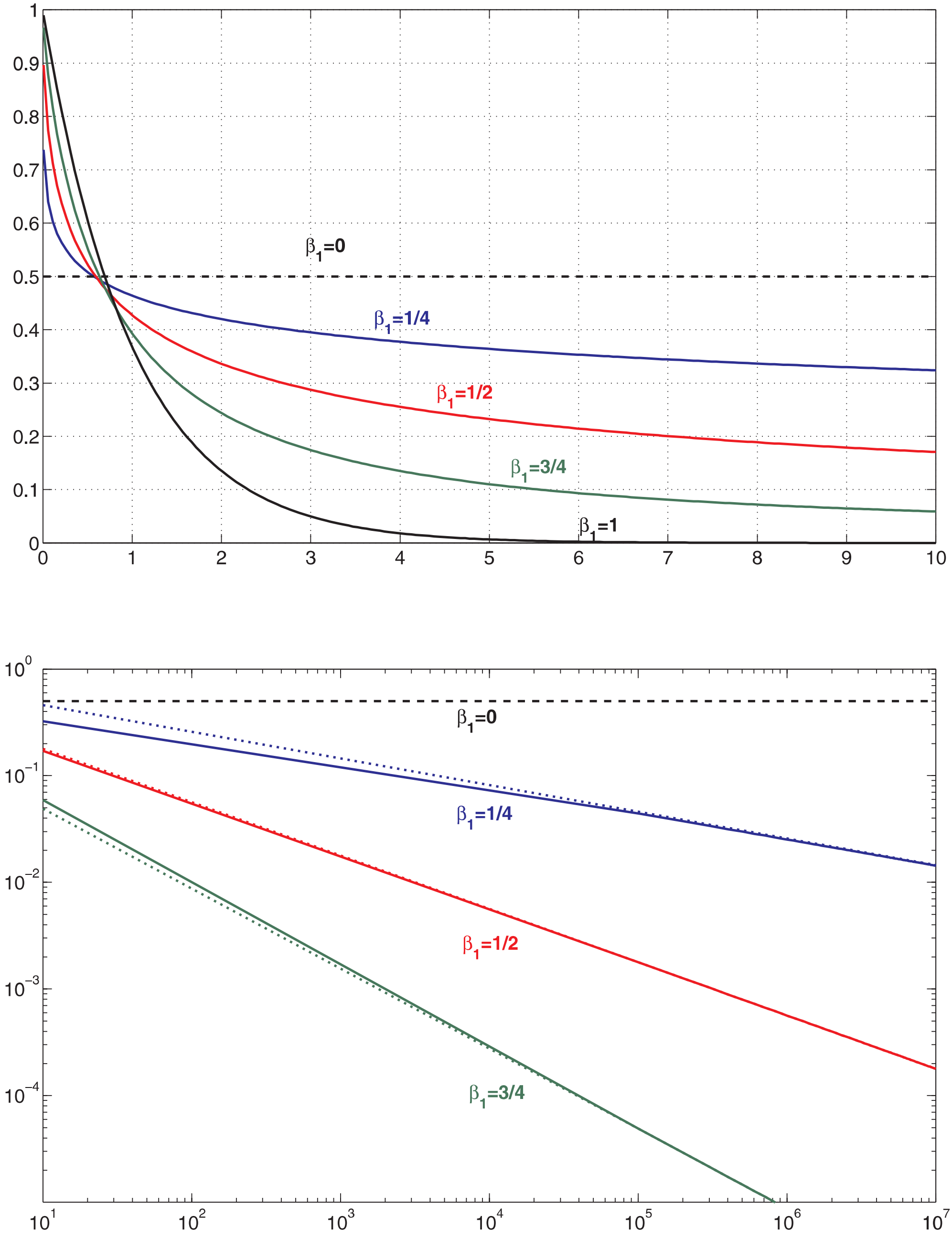

Figure 1. Fundamental solutions of the fractional relaxation of a single order $\beta_1 = 1/4, 1/2, 3/4, 1$. Top: linear scales; Bottom: logarithmic scales.

subjected to the initial condition $u(0^+) = u_*(0^+) = 1$, where

$$p(\beta) \geq 0, \quad \text{and} \quad \int_0^1 p(\beta) d\beta = c > 0. \tag{3.3}$$

The positive constant c can be taken as 1 if we want the integral to be normalized. For the weight function $p(\beta)$ we conveniently require that its primitive $P(\beta) = \int_0^\beta p(\beta')d\beta'$ vanishes at $\beta = 0$ and is there continuous from the right, attains the value c at $\beta = 1$ and has at most finitely many (upwards) jump points in the half-open interval $0 < \beta \leq 1$, these jump points allowing delta contributions to $p(\beta)$ (particularly relevant for discrete distributions of orders).

Since for distributed order the solution depends on the selected approach (as we shall show hereafter), we now distinguish the fractional equations (3.1) and (3.2) and their fundamental solutions by decorating in the Caputo case the variable $u(t)$ with subscript $*$.

As in Gorenflo and Mainardi (2006), the present analysis is based on the application of the Laplace transformation with particular attention to some special cases. Here, for these cases, we shall provide plots of the corresponding solutions.

3.2. The Integral Formula for the Fundamental Solutions

Let us now apply the Laplace transform to equations (3.1) and (3.2) by using the rules (A.15) and (A.13) appropriate to the R-L and C derivatives, respectively. Introducing the relevant functions

$$A(s) = s \int_0^1 p(\beta) s^{-\beta} d\beta, \tag{3.4}$$

and

$$B(s) = \int_0^1 p(\beta) s^{\beta} d\beta, \tag{3.5}$$

we then get for the R-L and C cases, after simple manipulation, the Laplace transforms of the corresponding fundamental solutions:

$$\widetilde{u}(s) = \frac{1}{s + \lambda A(s)}, \tag{3.6}$$

and

$$\widetilde{u}_*(s) = \frac{B(s)/s}{\lambda + B(s)}. \tag{3.7}$$

We note that in the particular case $p(\beta) = \delta(\beta - \beta_1)$ we have in (3.4): $A(s) = s^{1-\beta_1}$, and in (3.5): $B(s) = s^{\beta_1}$. Then, equations (3.6) and (3.7) provide the same result (2.5) as simple fractional relaxation.

By inverting the Laplace transforms in (3.6) and (3.7) we obtain the fundamental solutions for the R-L and C fractional relaxation of distributed order.

Let us start with the R-L derivatives. We get (by virtue of the Titchmarsh theorem on Laplace inversion) the representation

$$u(t) = -\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-rt} \mathrm{Im}\left\{\widetilde{u}\left(r\mathrm{e}^{i\pi}\right)\right\} dr, \tag{3.8}$$

which requires the expression of $-\mathrm{Im}\{1/[s + \lambda A(s)]\}$ along the ray $s = r\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function $s^{-\beta}$). We write

$$A\left(r\mathrm{e}^{i\pi}\right) = \rho\cos(\pi\gamma) + i\rho\sin(\pi\gamma), \tag{3.9}$$

where

$$\begin{cases} \rho = \rho(r) = \left|A(r\mathrm{e}^{i\pi})\right|, \\ \gamma = \gamma(r) = \dfrac{1}{\pi}\arg\left[A(r\mathrm{e}^{i\pi})\right]. \end{cases} \tag{3.10}$$

Then, after simple calculations, we get

$$u(t) = \int_0^\infty \mathrm{e}^{-rt} H(r;\lambda) dr, \tag{3.11}$$

with

$$H(r;\lambda) = \frac{1}{\pi}\frac{\lambda\rho\sin(\pi\gamma)}{r^2 - 2\lambda r\rho\cos(\pi\gamma) + \lambda^2\rho^2} \geq 0. \tag{3.12}$$

Similarly for the C derivatives we obtain

$$u_*(t) = -\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-rt} \mathrm{Im}\left\{\widetilde{u}_*\left(r\mathrm{e}^{i\pi}\right)\right\} dr, \tag{3.13}$$

which requires the expression of $-\mathrm{Im}\{B(s)/[s(\lambda + B(s))]\}$ along the ray $s = r\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function s^{β}). We write

$$B\left(r\mathrm{e}^{i\pi}\right) = \rho_*\cos(\pi\gamma_*) + i\rho_*\sin(\pi\gamma_*), \tag{3.14}$$

where

$$\begin{cases} \rho_* = \rho_*(r) = \left|B(r\mathrm{e}^{i\pi})\right|, \\ \gamma_* = \gamma_*(r) = \dfrac{1}{\pi}\arg\left[B(r\mathrm{e}^{i\pi})\right]. \end{cases} \tag{3.15}$$

After simple calculations we get

$$u_*(t) = \int_0^\infty \mathrm{e}^{-rt} K(r;\lambda)\,dr, \tag{3.16}$$

with

$$K(r;\lambda) = \frac{1}{\pi r}\frac{\lambda\rho_*\sin(\pi\gamma_*)}{\lambda^2 + 2\lambda\rho_*\cos(\pi\gamma_*) + \rho_*^2} \geq 0. \tag{3.17}$$

We note from (3.11) and (3.16) that, since $H(r;\lambda)$ and $K(r;\lambda)$ are non-negative functions of r for any $\lambda \in \mathbb{R}^+$, the fundamental solutions $u(t)$ and $u_*(t)$ keep the relevant property to be *completely monotone*.

The integral expressions (3.11) and (3.16) provide a sort of spectral representation of the fundamental solutions that will be used to numerically evaluate these solutions in some examples considered as interesting cases.

Furthermore, it is quite instructive to compute for the fundamental solutions their asymptotic expressions for $t \to 0$ and $t \to \infty$ because they provide analytical (even if approximated) representations for sufficiently short and long time respectively, and useful checks on the numerical evaluation in the above time ranges.

To derive these asymptotic representations we shall apply the Tauberian theory of Laplace transforms. According to this theory the asymptotic behaviour of a function $f(t)$ near $t = \infty$ and $t = 0$ is (formally) obtained from the asymptotic behaviour of its Laplace transform $\widetilde{f}(s)$ for $s \to 0^+$ and for $s \to +\infty$, respectively. For this purpose we note the asymptotic representations, from (3.6):

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{1}{\lambda A(s)}, & s \to 0^+, \quad \text{being} \quad A(s)/s >> \lambda, \\ \dfrac{1}{s}\left[1 - \lambda\dfrac{A(s)}{s}\right], & s \to +\infty, \quad \text{being} \quad A(s)/s << 1/\lambda, \end{cases} \tag{3.18}$$

and from (3.7):

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{1}{\lambda}\dfrac{B(s)}{s}, & s \to 0^+, \quad \text{being} \quad B(s) << \lambda, \\ \dfrac{1}{s}\left[1 - \dfrac{\lambda}{B(s)}\right], & s \to +\infty, \quad \text{being} \quad B(s) >> \lambda. \end{cases} \tag{3.19}$$

4. EXAMPLES

Since finding explicit solution formulas is not possible for the relaxation equations (3.1) and (3.2) we shall concentrate our interest on some typical choices for the weight function $p(\beta)$ in (3.3) that characterizes the order distribution. For these choices we present the numerical evaluation of the Titchmarsh integral formula, see equations (3.8)–(3.12) for $u(t)$ (the R-L case), and equations (3.13)–(3.17) for $u_*(t)$ (the C case). The numerical results are checked by verifying the matching with the asymptotic expressions for $u(t)$ and $u_*(t)$ as $t \to 0$ and $t \to +\infty$, obtained via the Tauberian theory for Laplace transforms, according to equations (3.18) and (3.19).

4.1. Double-order Fractional Relaxation

We now consider the choice

$$p(\beta) = p_1\delta(\beta - \beta_1) + p_2\delta(\beta - \beta_2), \quad 0 < \beta_1 < \beta_2 \le 1, \tag{4.1}$$

where the constants p_1 and p_2 are both positive, conveniently restricted to the normalization condition $p_1 + p_2 = 1$. Then for the R-L case we have

$$A(s) = p_1 s^{1-\beta_1} + p_2 s^{1-\beta_2}, \tag{4.2}$$

so that, inserting (4.2) in (3.6),

$$\widetilde{u}(s) = \frac{1}{s[1 + \lambda(p_1 s^{-\beta_1} + p_2 s^{-\beta_2})]}, \tag{4.3}$$

Similarly, for the C case we have

$$B(s) = p_1 s^{\beta_1} + p_2 s^{\beta_2}, \tag{4.4}$$

so that, inserting (4.3) in (3.7),

$$\widetilde{u}_*(s) = \frac{p_1 s^{\beta_1} + p_2 s^{\beta_2}}{s[\lambda + p_1 s^{\beta_1} + p_2 s^{\beta_2}]}. \tag{4.5}$$

We leave as an exercise the derivation of the spectral functions $H(r;\lambda)$ and $K(r;\lambda)$ of the corresponding fundamental solutions that are used for the numerical computation. The numerical results are checked by their matching with the asymptotic expressions that we evaluate by invoking the Tauberian theory and using equations (3.18) and (3.19) jointly with equations (4.2) and (4.3) respectively.

For the R-L-case we note that in (4.2) $s^{1-\beta_1}$ is negligibly small in comparison with $s^{1-\beta_2}$ for $s \to 0^+$ and, vice versa, $s^{1-\beta_2}$ is negligibly small in comparison to $s^{1-\beta_1}$ for $s \to +\infty$. Similarly for the C-case we note that in (4.3) s^{β_2} is negligibly small in comparison to s^{β_1} for $s \to 0^+$ and, vice versa, s^{β_1} is negligibly small in comparison s^{β_2} for $s \to +\infty$.

As a consequence of these considerations we get for the R-L case, if $\beta_2 < 1$,

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{1}{\lambda p_2} s^{\beta_2 - 1}, & s \to 0^+, \\ \dfrac{1}{s}(1 - \lambda p_1 s^{-\beta_1}), & s \to +\infty, \end{cases} \tag{4.6}$$

so that

$$u(t) \sim \begin{cases} \dfrac{1}{\lambda p_2} \dfrac{t^{-\beta_2}}{\Gamma(1-\beta_2)}, & t \to +\infty, \\ 1 - \lambda p_1 \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)}, & t \to 0^+. \end{cases} \tag{4.7}$$

We note that the equation (4.5a) and henceforth equation (4.6a) lose their meaning for $\beta_2 = 1$. In this case we need a more careful reasoning: we consider the expression for $s \to 0$ provided by (3.18) as it stands, that is

$$\widetilde{u}(s) \sim \frac{1}{\lambda[p_1 s^{1-\beta_1} + p_2]} = \frac{1}{\lambda p_1} \frac{1}{s^{1-\beta_1} + p_2/(\lambda p_1)}. \tag{4.8}$$

By virtue of the Laplace transform pair

$$t^{\nu-1} E_{\mu,\nu}\left(-q t^{\mu}\right) \div \frac{s^{\mu-\nu}}{s^{\mu} + q}, \tag{4.9}$$

see equation (1.80) in Podlubny (1999), where $E_{\mu,\nu}$ denotes the Mittag–Leffler function in two parameters[5] we get, with $q = p_2/(\lambda p_1)$ and $\mu = \nu = 1 - \beta_1$, as $t \to +\infty$:

$$u(t) \sim \frac{1}{\lambda p_1} t^{-\beta_1} E_{1-\beta_1,1-\beta_1}\left(-q t^{1-\beta_1}\right) = -\frac{1}{\lambda p_1} \frac{d}{dt} E_{1-\beta_1}\left(-q t^{1-\beta_1}\right). \tag{4.10}$$

Taking into account the asymptotic behaviour of the Mittag-Leffler function, we finally get

$$u(t) \sim \lambda \frac{p_1}{p_2} \frac{1-\beta_1}{\Gamma(\beta_1)} t^{-(2-\beta_1)} \quad \text{as} \quad t \to +\infty. \tag{4.11}$$

Similarly for the C case we get:

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{p_1}{\lambda} s^{\beta_1 - 1}, & s \to 0^+, \\ \dfrac{1}{s}\left(1 - \dfrac{\lambda}{p_2} s^{-\beta_2}\right), & s \to +\infty, \end{cases} \tag{4.12}$$

so that

$$u_*(t) \sim \begin{cases} \dfrac{p_1}{\lambda} \dfrac{t^{-\beta_1}}{\Gamma(1-\beta_1)}, & t \to +\infty, \\ 1 - \dfrac{\lambda}{p_2} \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)}, & t \to 0^+. \end{cases} \tag{4.13}$$

We exhibit in Figures 2 and 3 plots of the fundamental solutions for R-L and C fractional relaxation, respectively, for some $\{\beta_1, \beta_2\}$ combinations: {1/8, 1/4}; {1/4, 1/2}; {1/2, 3/4};

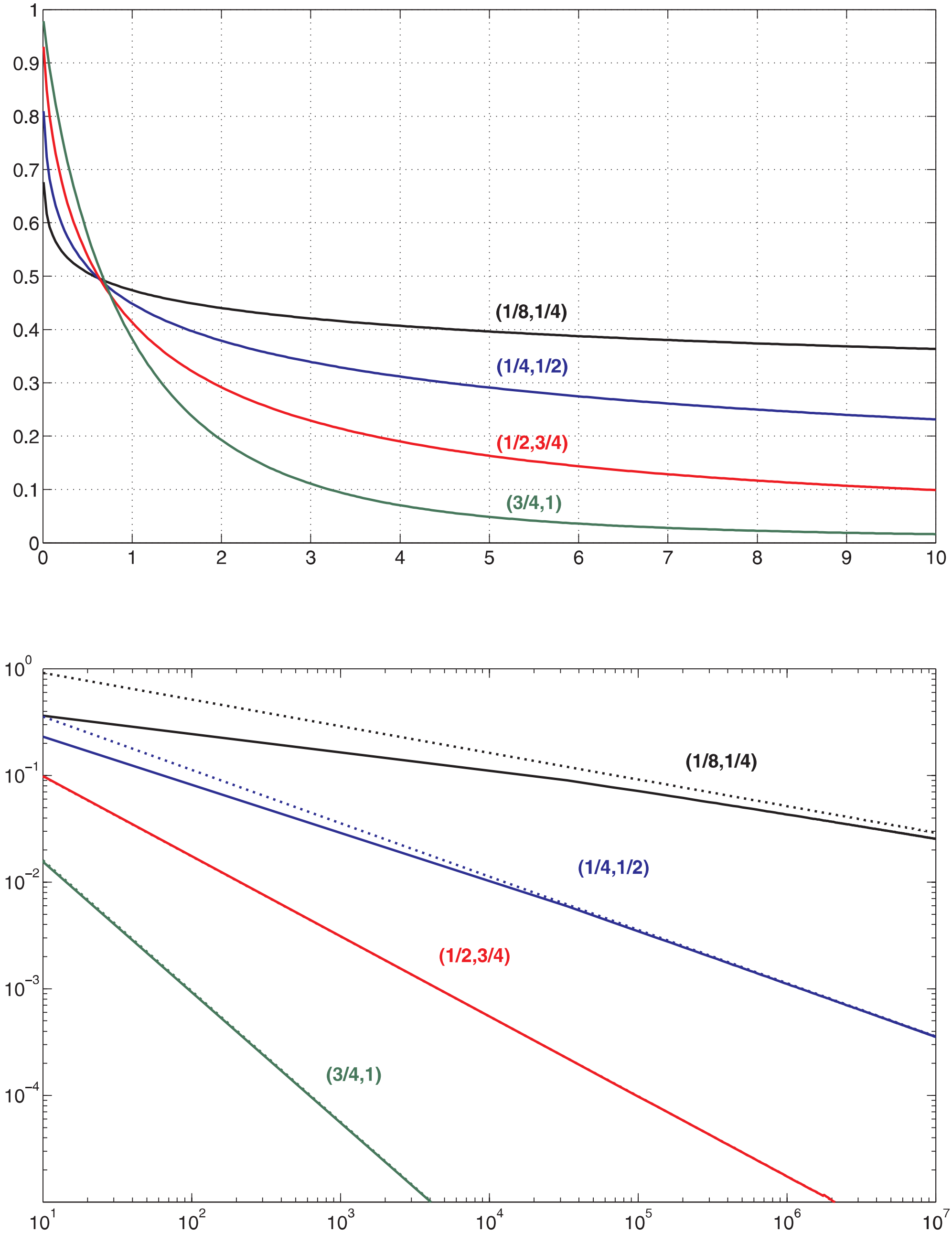

Figure 2. Fundamental solutions of the R-L fractional relaxation of double order in some $\{\beta_1, \beta_2\}$ combinations: $\{1/8, 1/4\}$; $\{1/4, 1/2\}$; $\{1/2, 3/4\}$; $\{3/4, 1\}$. Top: linear scales; Bottom: logarithmic scales.

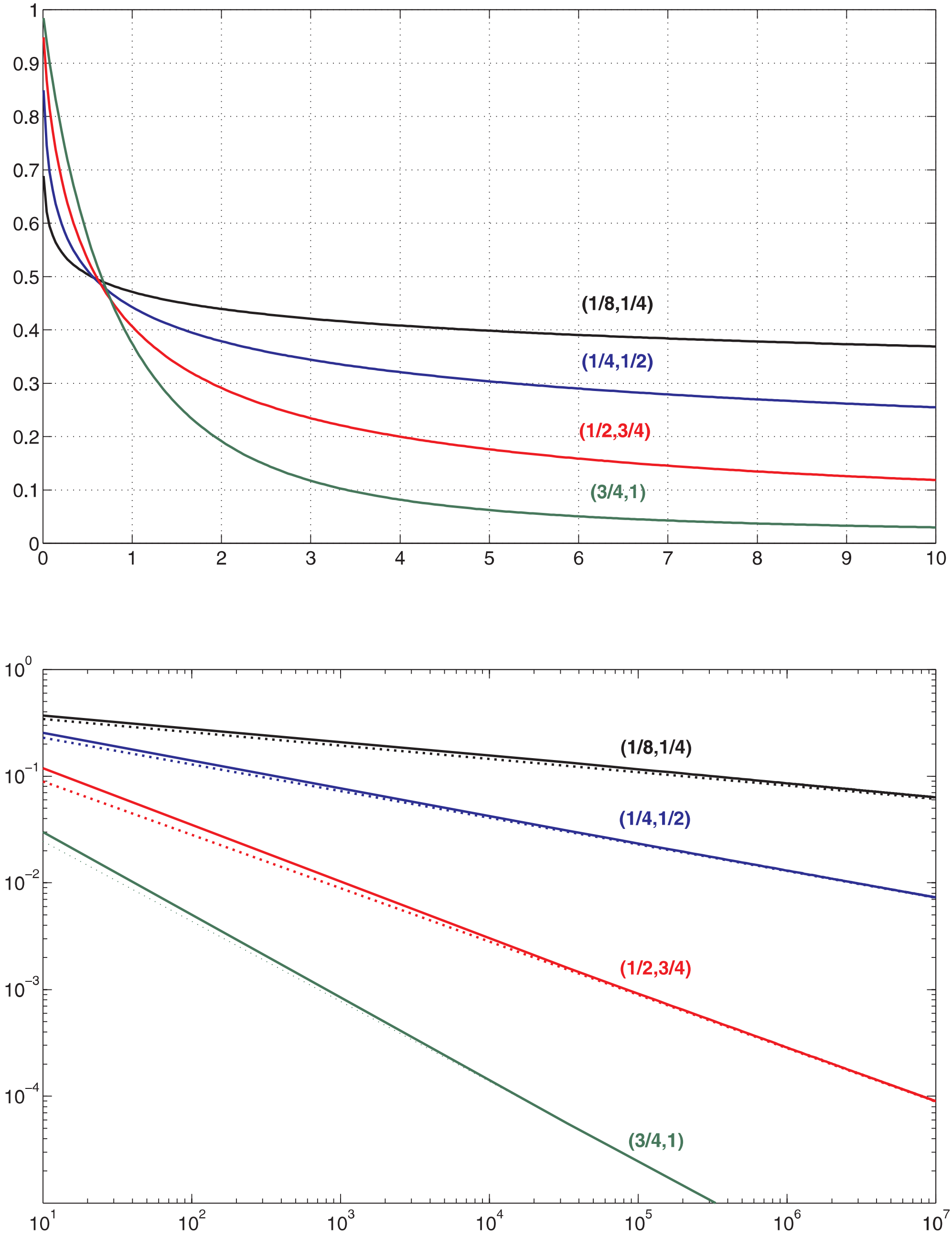

Figure 3. Fundamental solutions of the C-fractional relaxation of double order in some $\{\beta_1, \beta_2\}$ combinations: $\{1/8, 1/4\}$; $\{1/4, 1/2\}$; $\{1/2, 3/4\}$; $\{3/4, 1\}$. Top: linear scales; Bottom: logarithmic scales.

$\{3/4, 1\}$. We have chosen $p_1 = p_2 = 1/2$ and, as usual $\lambda = 1$. From the plots the reader is expected to verify the role played by the different orders for small and large times according to the corresponding asymptotic expressions; see equations (4.6), (4.9), (4.10) and (4.12).

4.2. Uniformly Distributed Order Fractional Relaxation

We now consider the choice

$$p(\beta) = 1, \quad 0 < \beta < 1. \tag{4.14}$$

For the R-L case we have

$$A(s) = s \int_0^1 s^{-\beta} d\beta = \frac{s-1}{\log s}, \tag{4.15}$$

hence, inserting (4.14) in (3.6)

$$\widetilde{u}(s) = \frac{\log s}{s \log s + \lambda(s-1)}. \tag{4.16}$$

For the C case we have

$$B(s) = \int_0^1 s^{\beta} d\beta = \frac{s-1}{\log s}, \tag{4.17}$$

hence, inserting (4.16) in (3.7),

$$\widetilde{u}_*(s) = \frac{1}{s} \frac{s-1}{\lambda \log s + s - 1} = \frac{1}{s} - \frac{1}{s} \frac{\lambda \log s}{\lambda \log s + s - 1}. \tag{4.18}$$

We note that for this special order distribution we have $A(s) = B(s)$ but the corresponding fundamental solutions are quite different, as we see from their Laplace transforms (4.15) and (4.17).

Then, invoking the Tauberian theory for *regularly varying functions* (power functions multiplied by *slowly varying functions*[6]), a topic adequately treated in the treatise on Probability by Feller (1971, Chapter XIII.5), we have the following asymptotic expressions for the R-L and C cases.

For the R-L case we get

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{\log s}{\lambda(s-1)}, & s \to 0^+, \\[2ex] \dfrac{1}{s}\left[1 - \lambda \dfrac{s-1}{s \log s}\right], & s \to +\infty, \end{cases} \tag{4.19}$$

so

$$u(t) \sim \begin{cases} \dfrac{1}{\lambda} \mathrm{e}^t \mathcal{E}_1(t) \sim \dfrac{1}{\lambda t}, & t \to +\infty, \\ 1 - \dfrac{\lambda}{|\log(1/t)|}, & t \to 0^+. \end{cases} \tag{4.20}$$

In (4.19) $\mathcal{E}_1(t) := \int_t^\infty \frac{\mathrm{e}^{-u}}{u} du$ denotes the exponential integral; see Abramowitz and Stegun (1965, Ch. 5 and the Laplace transform pair (29.3.100)).

For the C case we get

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{1}{\lambda s \log(1/s)}, & s \to 0^+, \\ \dfrac{1}{s} - \dfrac{\lambda \log s}{s^2}, & s \to +\infty, \end{cases} \tag{4.21}$$

so

$$u_*(t) \sim \begin{cases} \dfrac{1}{\lambda \log t}, & t \to +\infty, \\ 1 - \lambda t \log(1/t), & t \to 0^+. \end{cases} \tag{4.22}$$

In Figure 4 we display plots of the fundamental solutions for R-L and C uniformly distributed fractional relaxation, adopting as previously, in the top figure, linear scales ($0 \le t \le 10$), and in the lower figure, logarithmic scales ($10^1 \le t \le 10^7$).

For comparison in the top diagram the plots for single orders $\beta_1 = 0, 1/2, 1$ are shown. We note that for $1 < t < 10$ the R-L and C plots are close to that for $\beta_1 = 1/2$ from above and from below, respectively.

In the bottom figure (where the plot for β_1 is not visible because of its faster exponential decay) we have added in dotted lines the asymptotic solutions for large times. We recognize that the C plot is decaying much more slowly than any power law whereas the R-L plot is decaying as t^{-1}; this means that for large times these plots are the border lines for all plots corresponding to single order relaxation with $\beta_1 \in (0, 1)$.

5. CONCLUSIONS

We have investigated the relaxation equation with (discretely or continuously) distributed order of fractional derivatives both in the Riemann–Liouville and in the Caputo sense. Such equations can be seen as simple models of more general distributed order fractional evolution in a Banach space where the relaxation parameter λ is replaced by an operator A acting in this space. A relevant example is time-fractional diffusion where in the linear case the individual modes exhibit fractional relaxation. Our interest is focused on structural properties of the solutions, in particular on asymptotic behaviour at small and large times. In both approaches we find that the smallest order of occurring fractional differentiation determines the behaviour near infinity, but the largest order the behaviour near zero, in analogy to the special form of time-fractional diffusion explicitly governed by the distributed order deriv-

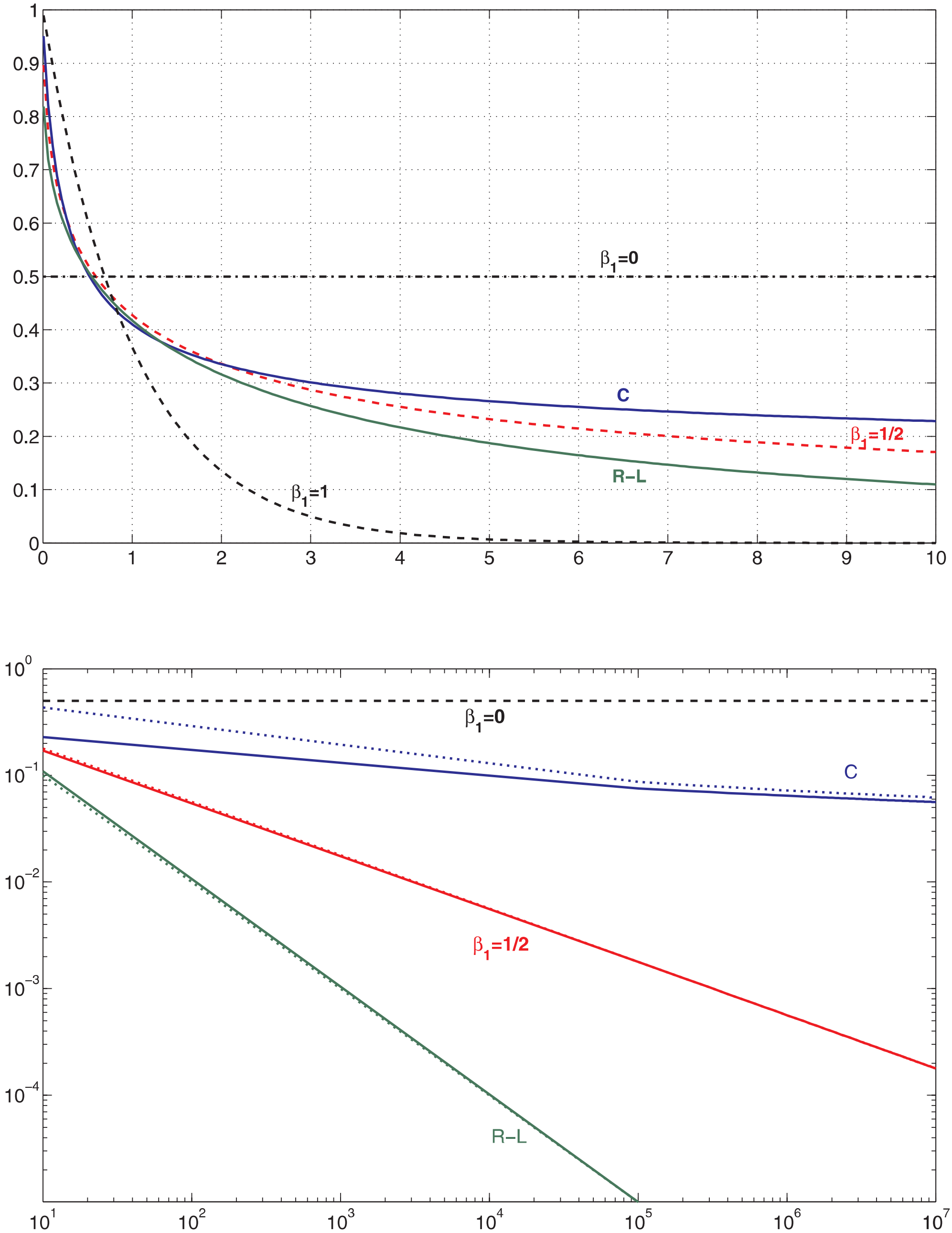

Figure 4. Fundamental solutions for R-L and C uniformly distributed fractional relaxation in comparison with some solutions for single orders. Top: linear scales; Bottom: logarithmic scales.

ative as in Chechkin et al. (2002a, 2002b) and in Langlands (2006), Mainardi et al. (2007) and Sokolov et al. (2004). We see that the two parameters β_1 and β_2 play opposite roles in our two cases (R-L) and (C). The topic deserves further study in several directions, e.g. in terms of integral transforms and special functions like those of Mittag–Leffler type.

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APPENDIX: ESSENTIALS OF FRACTIONAL CALCULUS

For a sufficiently well-behaved function $f(t)$ ($t \in \mathbb{R}^+$) we may define the fractional derivative of order μ ($m-1 < \mu \le m$, $m \in \mathbb{N}$), see Gorenflo and Mainardi (1997) and Podlubny (1999), in two different senses, that we refer here as to *Riemann–Liouville* (R-L) derivative and *Caputo* (C) derivative, respectively. Both derivatives are related to the so-called Riemann–Liouville fractional integral of order $\alpha > 0$ defined as

$$ {}_tJ^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0. \tag{A.1}$$

We recall the convention ${}_tJ^0 = I$ (Identity operator) and the semigroup property

$$ {}_tJ^\alpha \, {}_tJ^\beta = {}_tJ^\beta \, {}_tJ^\alpha = {}_tJ^{\alpha+\beta}, \quad \alpha, \beta \ge 0. \tag{A.2}$$

Furthermore

$$ {}_tJ^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \quad \alpha \ge 0, \quad \gamma > -1, \quad t > 0. \tag{A.3}$$

The fractional derivative of order $\mu > 0$ in the *Riemann–Liouville* sense is defined as the operator ${}_tD^\mu$ which is the left inverse of the Riemann–Liouville integral of order μ (in analogy with the ordinary derivative), that is

$$ {}_tD^\mu \, {}_tJ^\mu = I, \quad \mu > 0. \tag{A.4}$$

If m denotes the positive integer such that $m-1 < \mu \le m$, we recognize from equations (A.2) and (A.4) ${}_tD^\mu f(t) := {}_tD^m \, {}_tJ^{m-\mu} f(t)$, hence

$$ {}_tD^\mu f(t) = \begin{cases} \dfrac{d^m}{dt^m} \left[\dfrac{1}{\Gamma(m-\mu)} \int_0^t \dfrac{f(\tau)d\tau}{(t-\tau)^{\mu+1-m}} \right], & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{A.5}$$

For completion we define ${}_tD^0 = I$.

On the other hand, the fractional derivative of order $\mu > 0$ in the *Caputo* sense is defined as the operator ${}_tD_*^\mu$ such that ${}_tD_*^\mu f(t) := {}_tJ^{m-\mu} \, {}_tD^m f(t)$, hence

$$
{}_tD_*^\mu\, f(t) = \begin{cases} \dfrac{1}{\Gamma\,(m-\mu)} \int_0^t \dfrac{f^m(\tau)d\tau}{(t-\tau)^{\mu+1-m}}, & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{A.6}
$$

Thus, when the order is not integer the two fractional derivatives differ in that the derivative of order m does not generally commute with the fractional integral.

We point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than m , if its order μ is such that $m-1 < \mu \le m$. Furthermore we note that

$$
{}_tD^\mu\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)}\, t^{\gamma-\mu}, \quad \mu \ge 0, \quad \gamma > -1, \quad t > 0. \tag{A.7}
$$

Gorenflo and Mainardi (1997) have shown the essential relationships between the two fractional derivatives (when both of them exist),

$$
{}_tD_*^\mu f(t) = \begin{cases} {}_tD^\mu \left[f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \dfrac{t^k}{k!} \right], \\ {}_tD^\mu f(t) - \sum_{k=0}^{m-1} \dfrac{f^{(k)}(0^+)t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \quad m-1 < \mu < m. \tag{A.8}
$$

In particular, if $m = 1$ we have

$$
{}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu[f(t) - f(0^+)], \\ {}_tD^\mu\, f(t) - \dfrac{f(0^+)t^{-\mu}}{\Gamma(1-\mu)}, \end{cases} \quad 0 < \mu < 1. \tag{A.9}
$$

The *Caputo* fractional derivative, practically ignored in mathematical treatises, represents a sort of regularization in the time origin for the *Riemann–Liouville* fractional derivative. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim_{t\to 0^+} f(t)$ are required to be finite for $k = 0, 1, 2.\ldots m-1$.

We observe different behaviour of the two fractional derivatives at the end points of the interval $(m-1, m)$ namely when the order is any positive integer: whereas ${}_tD^\mu$ is, with respect to its order μ , an operator continuous at any positive integer, ${}_tD_*^\mu$ is an operator left-continuous since

$$
\begin{cases} \lim_{\mu\to(m-1)^+}\ {}_tD_*^\mu f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+), \\ \lim_{\mu\to m^-}\ {}_tD_*^\mu f(t) = f^{(m)}(t). \end{cases} \tag{A.10}
$$

We also note for $m-1 < \mu \le m$,

$$ {}_tD^{\mu} f(t) \; = \; {}_tD^{\mu} g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j t^{\mu-j}, \tag{A.11} $$

$$ {}_tD_*^{\mu} f(t) \; = \; {}_tD_*^{\mu} g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j t^{m-j}. \tag{A.12} $$

In these formulae the coefficients c_j are arbitrary constants. Last but not least, we point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation, according to which

$$ \mathcal{L}\left\{{}_tD_*^{\mu} f(t); s\right\} = s^{\mu} \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k} f^{(k)}(0^+), \quad m-1 < \mu \le m\,, \tag{A.13} $$

where $\widetilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} \mathrm{e}^{-st} f(t)\, dt\,,\ s \in \mathbb{C}$, and $f^{(k)}(0^+) := \lim_{t\to 0^+} f(t)$. The corresponding rule for the Riemann–Liouville derivative is more cumbersome: for $m-1 < \mu \le m$ it reads

$$ \mathcal{L}\left\{{}_tD^{\mu} f(t); s\right\} = s^{\mu} \widetilde{f}(s) - \sum_{k=0}^{m-1} \left[{}_tD^k\, {}_tJ^{(m-\mu)}\right] f(0^+) s^{m-1-k}, \tag{A.14} $$

where, in analogy with (A.13), the limit for $t \to 0^+$ is understood to be taken after the operations of fractional integration and derivation. As soon as all the limiting values $f^{(k)}(0^+)$ are finite and $m-1 < \mu < m$, the formula (A.14) simplifies into

$$ \mathcal{L}\left\{{}_tD^{\mu} f(t); s\right\} = s^{\mu} \widetilde{f}(s). \tag{A.15} $$

In the special case $f^{(k)}(0^+) = 0$ for $k = 0, 1, m-1$, we recover the identity between the two fractional derivatives, consistently with equation (A.8).

For more details on the theory and applications of fractional calculus we recommend consulting, in addition to the well-known books by Samko et al. (1993), by Miller and Ross (1993), and by Podlubny (1999), those that have appeared in recent years by Kilbas et al. (2006), by West et al. (2003), and by Zaslavsky (2005).

NOTES

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2. Both equations (2.3) and (2.4) are equivalent to the Volterra integral equation (of fractional type)

$$ u(t) = u(0^+) - \lambda\, {}_tJ^{\beta} u(t). $$

For example, we derive the R-L equation (2.3) from the fractional integral equation simply differentiating both sides of the latter, whereas we derive the fractional integral equation from the C equation (2.4) by fractional integration of order β. In fact, in view of the semigroup property (A.2) of the fractional integral, we note that

$$ {}_tJ^{\beta}\,{}_tD_*^{\beta}u(t) = {}_tJ^{\beta}\,{}_tJ^{1-\beta}\,{}_tD^{1}u(t) = {}_tJ^{1}\,{}_tD^{1}u(t) = u(t) - u(0^+). $$

In the limit $\beta = 1$ we recover the relaxation equation (2.1) with the solution (2.2). The reader interested in having more details on the two forms of fractional relaxation may consult, for the R-L approach, Hilfer (2000) and Nonnenmacher and Metzler (1995), and for the C approach, Caputo and Mainardi (1971), Gorenflo and Mainardi (1997) and Mainardi (1996).

3. Let us recall that the Mittag–Leffler function $E_\beta(z)$ ($\beta > 0$) is an entire transcendental function of order $1/\beta$, defined in the complex plane by the power series

$$ E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad \beta > 0, \quad z \in \mathbf{C}. $$

For details on it we refer the reader to Erdelyi et al. (1955), Kilbas et al. (2006), Gorenflo and Mainardi (1997), Podlubny (1999) and Samko et al. (1993). We remark that for $t \geq 0$ the function $E_\beta(-\lambda t^\beta)$ preserves the *complete monotonicity* of the exponential $\exp(-\lambda t)$: indeed it is represented in terms of a real Laplace transform of a non-negative function,

$$ E_\beta(-\lambda t^\beta) = \frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-rt}}{r} \frac{\lambda r^\beta \sin(\beta\pi)}{\lambda^2 + 2\lambda r^\beta \cos(\beta\pi) + r^{2\beta}} dr, \quad t \geq 0, \quad 0 < \beta < 1. $$

However it decreases at $t \to \infty$ like a power with exponent $-\beta$: $E_\beta(-\lambda t^\beta) \sim t^{-\beta}/[\lambda\Gamma(1-\beta)]$. If $\beta = 1/2$ we have for $t \geq 0$: $E_{1/2}(-\lambda\sqrt{t}) = \mathrm{e}^{\lambda^2 t}\mathrm{erfc}(\lambda\sqrt{t}) \sim 1/(\lambda\sqrt{\pi t})$ as $t \to \infty$, where erfc denotes the *complementary error* function; see Abramowitz and Stegun (1965).

4. We find a former idea of fractional derivative of distributed order in time in the 1969 book by Caputo, which was later developed by Caputo himself (Caputo, 1995; 2001) and by Bagley and Torvik (2000). A basic framework for the numerical solution of distributed-order differential equations has been recently introduced by Diethelm and Ford (2001) and Diethelm and Luchko (2004) and by Lorenzo and Hartley (2002) and Hartley and Lorenzo (2003).

5. The Mittag–Leffler function $E_{\mu,\nu}(z)$ ($\Re\{\mu\} > 0$, $\nu \in \mathbf{C}$) is defined by the power series

$$ E_{\mu,\nu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}, \quad z \in \mathbf{C}. $$

It generalizes the classical Mittag–Leffler function to which it reduces for $\nu = 1$. It is an entire transcendental function of order $1/\Re\{\mu\}$ on which the reader can gain more information by again consulting Erdelyi et al. (1955), Kilbas et al. (2006), Gorenflo and Mainardi (1997), Podlubny (1999) and Samko et al. (1993). With $\mu, \nu \in \mathbf{R}$ the function $E_{\mu,\nu}(-x)$ ($x \geq 0$) becomes a completely monotonic function of x if $0 < \mu \leq 1$ and $\nu \geq \mu$; see Miller and Samko (2001). This property is still valid when $x = qt^\mu$ ($q > 0$). In particular, for $0 < \mu = \nu < 1$ we note

$$ qt^{-(1-\mu)} E_{\mu,\mu}(-qt^\mu) = -\frac{d}{dt} E_\mu(-qt^\mu) \sim \frac{\mu}{q\Gamma(1-\mu)} t^{-(\mu+1)}, \quad t \to +\infty. $$

6. **Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, *slowly varying at zero* if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive

function $b(y)$, defined in a neighbourhood of infinity, *slowly varying at infinity* if $b(cy)/b(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $|\log y|^\gamma$ with $\gamma \in \mathbf{R}$.

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