



# Discrete random walk models for symmetric Lévy–Feller diffusion processes

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## Abstract

We propose a variety of models of random walk, discrete in space and time, suitable for simulating stable random variables of arbitrary index  $\alpha$  ( $0 < \alpha \leq 2$ ), in the symmetric case. We show that by properly scaled transition to vanishing space and time steps our random walk models converge to the corresponding continuous Markovian stochastic processes which we refer to as Lévy–Feller diffusion processes. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

By a Lévy–Feller diffusion process we mean a Markovian process governed by a stable probability density function (pdf) evolving in time,  $g_\alpha(x, t; \theta)$ , whose spatial Fourier transform (the characteristic function) reads

$$\hat{g}_\alpha(\kappa, t; \theta) = \int_{-\infty}^{+\infty} e^{i\kappa x} g_\alpha(x, t; \theta) dx = \exp(-t|\kappa|^\alpha e^{i(\text{sign } \kappa)\theta\pi/2}), \tag{1.1}$$

where  $x, \kappa \in \mathbf{R}$ ,  $t > 0$ . The two relevant parameters,  $\alpha$ , called the *index of stability*, and  $\theta$  (related to the asymmetry), improperly referred to as the *skewness*, are real numbers

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subject to the conditions (see, e.g. [1])

$$0 < \alpha \leq 2; \quad |\theta| \leq \begin{cases} \alpha, & \text{if } 0 < \alpha \leq 1, \\ 2 - \alpha, & \text{if } 1 < \alpha \leq 2. \end{cases} \quad (1.2)$$

By introducing the similarity variable  $xt^{-1/\alpha}$ , we can write  $g_\alpha(x, t; \theta) = t^{-1/\alpha} p_\alpha(xt^{-1/\alpha}; \theta)$ , where  $p_\alpha(x; \theta)$  is the stable pdf at  $t=1$ . The specific form of the characteristic function (1.1) allows us to recognize  $g_\alpha(x, t; \theta)$  as the Green function (fundamental solution) of the Cauchy problem

$$\frac{\partial}{\partial t} u(x, t) = D_\theta^\alpha [u(x, t)], \quad u(x, 0) = \delta(x), \quad x \in \mathbf{R}, \quad t > 0, \quad (1.3)$$

where  $D_\theta^\alpha$  is the pseudo-differential operator with symbol

$$\widehat{D}_\theta^\alpha = -|\kappa|^\alpha e^{i(\text{sign } \kappa)\theta\pi/2}. \quad (1.4)$$

Let us recall that a generic pseudo-differential operator  $A$ , acting with respect to the variable  $x \in \mathbf{R}$ , is defined through its Fourier representation, namely  $\int_{-\infty}^{+\infty} e^{i\kappa x} A[\phi(x)] dx = \widehat{A}(\kappa) \widehat{\phi}(\kappa)$ , where  $\phi(x)$  denotes a sufficiently well-behaved function in  $\mathbf{R}$ , and  $\widehat{A}(\kappa)$  is referred to as symbol of  $A$ , given as  $\widehat{A}(\kappa) = (Ae^{-i\kappa x})e^{+i\kappa x}$ .

With the names of Lévy and Feller we have intended to honour both Paul Lévy [4] who first introduced the class of stable distributions, see [2–4] and William Feller [5] who first investigated the semigroups generated by a pseudo-differential equation of type (1.3)–(1.4). For  $\alpha=2$  and  $\alpha=1$  (with  $\theta=0$ ) we recover the standard Gaussian and Cauchy pdf's

$$g_2(x, t; 0) = \frac{1}{2\sqrt{\pi}} t^{-1/2} \exp\left(-\frac{x^2}{4t}\right), \quad g_1(x, t; 0) = \frac{1}{\pi} \frac{t}{x^2 + t^2}. \quad (1.5)$$

In physics, the first recognition that the (symmetric) Lévy distribution could be characterized via a pseudo-differential operator of type (1.3)–(1.4) was made explicitly by West and Seshadri [6]. Recently, Gorenflo and Mainardi [7–9] have revised Feller's original arguments by interpreting (1.3) as a space-fractional diffusion equation (of order  $\alpha$  and “skewness”  $\theta$ ) and have provided a variety of related random walk models, discrete in space and time, which by properly scaled transition to vanishing space and time steps converge to the corresponding continuous Lévy–Feller processes. In other words the discrete probability distributions generated by the random walk models have been proved to belong to the domain of attraction of the corresponding stable distribution.

Here, limiting ourselves to the symmetric case ( $\theta=0$ ), we present the main features of the random walk models by Gorenflo and Mainardi, and we display preliminary results of a few numerical case studies, which can be of some interest in econophysics. The Lévy statistics in modelling fluctuations of economical and financial variables, formerly used by Mandelbrot in the early sixties, is still adopted with success, see, e.g. [10–12].

## 2. Outline of the general theory

Let  $Y$  be an integer-valued random variable and let the random variables  $Y_1, Y_2, Y_3, \dots$  be iid (= independent identically distributed), all having their probability distribution common with  $Y$ . We define a spatial-temporal grid  $\{(x_j, t_n) | j \in \mathbf{Z}, n \in \mathbf{N}_0\}$  by  $x_j = x_j(h) = jh, t_n = t_n(\tau) = n\tau$ , where  $h > 0$  and  $\tau > 0$ . Then we consider the sequence of random variables

$$S_n = hY_1 + hY_2 + \dots + hY_n, \quad n \in \mathbf{N}, \tag{2.1}$$

with (for convenience)  $S_0 = 0$ , and interpret it as follows. A particle, sitting in  $x = x_0 = 0$  at time  $t = t_0 = 0$  finds itself at a later instant  $t = t_n$  in point  $x = S_n$  which is an integer multiple of  $h$ . We recognize the  $p_k = P(Y = k)$  (for  $k \in \mathbf{Z}$ ) as “transition probabilities”:  $p_k$  is the probability of a particle jumping from a point  $x_j = S_n$  to a point  $x_{j+k} = S_{n+1}$  as time proceeds from  $t_n$  to  $t_{n+1}$ . All  $p_k$  are non-negative, and their sum equals 1.

The probability  $y_j(t_n)$  of sojourn of our particle in point  $x_j$  at instant  $t_n$  obeys the *transition law*

$$y_j(t_{n+1}) = \sum_{k=-\infty}^{+\infty} p_k y_{j-k}(t_n), \quad y_j(0) = \delta_{j0}, \quad j \in \mathbf{Z}, n \in \mathbf{N}_0 \tag{2.2}$$

which has the form of a discrete convolution. Hence, introducing *generating functions*

$$\tilde{p}(z) = \sum_{j=-\infty}^{+\infty} p_j z^j, \quad \tilde{y}_n(z) = \sum_{j=-\infty}^{+\infty} y_j(t_n) z^j, \tag{2.3}$$

we obtain

$$\tilde{y}_n(z) = \tilde{y}_0(z) [\tilde{p}(z)]^n = [\tilde{p}(z)]^n, \quad n \in \mathbf{N}_0. \tag{2.4}$$

The power series in (2.3) and (2.4) are absolutely and uniformly convergent on  $|z| = 1$  and assume the value 1 at  $z = 1$ . Putting  $z = e^{i\kappa h}, \kappa \in \mathbf{R}$ , and observing  $z^j = e^{i\kappa j h} = e^{i\kappa x_j}$ , we recognize  $\hat{p}(\kappa; h) = \tilde{p}(e^{i\kappa h})$  and  $\hat{y}(\kappa, t_n; h) = \tilde{y}_n(e^{i\kappa h})$  as *characteristic functions* of the random variables  $hY$  and  $S_n$ , respectively.

Our aim is to approximate the Lévy–Feller diffusion process, which is governed by the evolution equation (1.3), arbitrarily well. To this purpose we introduce a strictly monotonic *scaling relation*  $\tau = \sigma(h) \rightarrow 0$  as  $h \rightarrow 0$ . We will fix  $t > 0$  and let  $h$  (and likewise  $\tau$ ) go to zero over such values that always  $n (= t/\tau = t/\sigma(h))$  is a positive integer. Then we have the equivalences

$$n \rightarrow \infty \Leftrightarrow h \rightarrow 0 \Leftrightarrow \tau \rightarrow 0,$$

and  $h$  depends on  $\tau$ , finally on  $n$ , so that  $h = h(n)$ . Replacing  $h$  by  $h(n)$  in (2.1) we obtain a sequence of random variables  $X_n$  with characteristic functions  $\hat{y}(\kappa, t_n; h) = [\hat{p}(e^{i\kappa h})]^n$  (note that now  $t_n = t$  is fixed). Invoking Theorem 3.6.1 of Lukacs [13], what remains to be shown is that  $\hat{y}(\kappa, t; h) \rightarrow \exp(-t|\kappa|^\alpha)$  as  $h \rightarrow 0$ , the characteristic function of the corresponding symmetric Lévy–Feller process. For this it suffices that, for fixed  $\kappa \neq 0$ ,

$$\log[\hat{y}(\kappa, t; h)] \equiv \frac{t}{\sigma(h)} \log[\tilde{p}(e^{i\kappa h})] \rightarrow -t|\kappa|^\alpha \quad \text{as } h \rightarrow 0. \tag{2.5}$$

Our random walk can be interpreted as a “difference scheme” to approximate the evolution equation (1.3), if we write (2.2) in the equivalent form, observing the scaling relation,

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = \frac{1}{\sigma(h)} \left[ (p_0 - 1)y_j(t_n) + \sum_{k \neq 0} p_k y_{j-k}(t_n) \right]. \tag{2.6}$$

In fact, after division by the spatial mesh-width  $h$ , the LHS is the explicit discrete approximation to the first-order time derivative  $\partial u(x, t)/\partial t$ , and the RHS can be considered as a particular discrete approximation to the space pseudo-differential term  $D_0^\alpha[u(x, t)]$ , provided we mean  $y_j(t_n) = \int_{x_j-h/2}^{x_j+h/2} u(x, t_n) dx \approx hu(x_j, t_n)$  with  $y_j(0) = \delta_{j0}$ .

### 3. The random walk models

From the previous section we have learnt that, in order to construct discrete random walk models which are convergent (in distribution) to the (symmetric) stable pdf’s, the clue points are: (1) to guess a suitable *generating function*  $\tilde{p}(z)$ , whose coefficients of its power series expansion provide the transition probabilities, (2) to determine the corresponding *scaling relation*  $\tau = \sigma(h)$  which ensures the required convergence.

In the classical case of the Gaussian distribution ( $\alpha=2$ ) the matter is easily treated if we remember that the corresponding density is the fundamental solution of the standard diffusion equation, which is known to be well approximated via the finite-difference equation

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = \frac{y_{j+1}(t_n) - 2y_j(t_n) + y_{j-1}(t_n)}{h^2}, \quad y_j(0) = \delta_{j0}. \tag{3.1}$$

In this case, introducing the scaling parameter  $\mu = \tau/h^2$ , so  $\tau = \sigma(h) = \mu h^2$ , the transition probabilities turn out to be

$$p_0 = 1 - 2\mu, \quad p_{\pm 1} = \mu, \quad p_{\pm k} = 0, \quad k = 2, 3 \dots, \tag{3.2}$$

subject to the condition  $0 < \mu \leq 1/2$ . Thus the generating function is

$$\tilde{p}(z) = 1 + \mu[z - 2 + z^{-1}]. \tag{3.3}$$

The proof of the convergence to the Gaussian is simple since one easily finds  $[t/(\mu h^2)] \times \log[\tilde{p}(e^{i\kappa h})] \rightarrow -t\kappa^2$  as  $h \rightarrow 0$ . The scheme (3.2) means that for approximation of the standard Gaussian process the corresponding random walk model exhibits only jumps of one step to the right or one to the left or jumps of width zero. For the stable non Gaussian processes we expect to find a non-polynomial generating function with infinitely many transition coefficients which imply the occurrence of arbitrarily large jumps. It is common practice to refer to the corresponding random walks as to *Lévy flights*.

In the following, limiting ourselves to the symmetric cases ( $\theta = 0$ ), we shall resume the main features of three different random walk models, referred to as (RW1), (RW2), (RW3), of which Gorenflo and Mainardi have proved the convergence to the

corresponding continuous processes. For each model we give the generating function  $\tilde{p}(z)$  with the transition probabilities  $p_k$  and the scaling relation  $\tau = \sigma(h)$ , referring to the original papers for details. From the analysis of the classical Gaussian case we find it natural to first introduce the scaling parameter  $\mu = \tau/h^\alpha$  with  $0 < \alpha \leq 2$  but, as we shall show later, this will not necessarily imply  $\sigma(h) = \mu h^\alpha$ , with  $\mu$  constant for fixed  $\alpha$ .

The model (RW1) has been introduced and discussed in [7], starting from the identification of the operator  $D_0^\alpha$  in the framework of fractional calculus, see [5], and then applying, in the authors' original approach, the Grünwald–Letnikov discretized scheme. For this model we need to keep distinct the two cases (a)  $0 < \alpha < 1$  and (b)  $1 < \alpha \leq 2$ , the case  $\alpha = 1$  being excluded in this treatment. The generating function is

$$\tilde{p}(z) = \begin{cases} 1 - \frac{\mu}{2 \cos(\alpha\pi/2)} [(1-z)^\alpha + (1-z^{-1})^\alpha], & 0 < \alpha < 1, \\ 1 - \frac{\mu}{2 \cos(\alpha\pi/2)} [z^{-1}(1-z)^\alpha + z(1-z^{-1})^\alpha], & 1 < \alpha \leq 2. \end{cases} \quad (3.4)$$

For both cases the scaling relation is confirmed to be  $\tau = \sigma(h) = \mu h^\alpha$ , but the parameter  $\mu$  is subject to different restrictions to ensure that  $0 \leq p_0 < 1$ .

In the case (a) we have  $0 < \mu \leq \cos(\alpha\pi/2)$  and

$$\begin{cases} p_0 = 1 - \frac{\mu}{\cos(\alpha\pi/2)}, \\ p_{\pm k} = (-1)^{k+1} \frac{\mu}{2 \cos(\alpha\pi/2)} \binom{\alpha}{k} \quad \text{for } k = 1, 2, 3, \dots \end{cases} \quad (3.5a)$$

In the case (b) we have  $0 < \mu \leq |\cos(\alpha\pi/2)|/\alpha$  and

$$\begin{cases} p_0 = 1 - \frac{\mu\alpha}{|\cos(\alpha\pi/2)|}, \quad p_{\pm 1} = \frac{\mu}{2|\cos(\alpha\pi/2)|} \left[ \binom{\alpha}{2} + 1 \right], \\ p_{\pm k} = (-1)^{k+1} \frac{\mu}{2|\cos(\alpha\pi/2)|} \binom{\alpha}{k+1} \quad \text{for } k = 2, 3, 4, \dots \end{cases} \quad (3.5b)$$

We note that, whereas the classical Gaussian random walk (3.2) is promptly recovered from (3.5b) for  $\alpha = 2$ , the random walk for the Cauchy process ( $\alpha = 1$ ) cannot be obtained, neither directly nor by a passage to the limit  $\alpha \rightarrow 1$ . Indeed, in both the limits  $\alpha \rightarrow 1^-$  and  $\alpha \rightarrow 1^+$  the permissible range of the scaling factor  $\mu$  is vanishing. In numerical practice the consequence will be that if  $\alpha$  is near 1 the convergence is slow: for good approximation we will need a very small step-time  $\tau$  with respect to the step-length  $h$ .

Apart from (RW1) we shall show how the two other random walk models, (RW2) and (RW3), that we are going to briefly discuss, exhibit a smooth scaling law in the range  $0 < \alpha < 2$ , so the case  $\alpha = 1$  is no longer singular. However, “there is no free lunch”. We have to pay for the good behaviour at  $\alpha = 1$  with bad behaviour at  $\alpha = 2$  in a sense to be seen later.

The model (RW2) is obtained by imposing for the transition probabilities an expression in terms of binomial coefficients, as suggested from (3.5b), which is required to

be regular in the limit as  $\alpha \rightarrow 1$ , namely

$$p_0 = 1 - 2\lambda, \quad p_{\pm k} = (-1)^{k+1} \frac{\lambda}{\alpha - 1} \binom{\alpha}{k + 1} \quad \text{for } k = 1, 2, 3, \dots, \quad (3.6)$$

where  $\lambda$ , subject to the condition  $0 < \lambda \leq 1/2$ , is to be determined. The model has been extensively discussed in [9] whereas the particular case  $\alpha = 1$  (related to the Cauchy process) has been formerly presented in [8]. To guarantee the convergence for all  $\alpha$  ( $0 < \alpha \leq 2$ ) to the corresponding continuous process, the asymptotics as  $h \rightarrow 0$  still requires a scaling relation of the kind  $\tau = \mu h^\alpha$  (with  $\mu$  constant), namely

$$\tau = \sigma(h) = \begin{cases} \frac{2\lambda \cos(\alpha\pi/2)}{1 - \alpha} h^\alpha & \text{if } 0 < \alpha < 2, \alpha \neq 1, \\ \lambda\pi h & \text{if } \alpha = 1, \\ \lambda h^2 & \text{if } \alpha = 2. \end{cases} \quad (3.7)$$

We recognize that for  $0 < \alpha < 2$  the model (RW2) exhibits a smooth scaling law but a discontinuity is present at  $\alpha = 2$  as can be seen by taking the limit as  $\alpha \rightarrow 2^-$ . This allows us to recover for  $\alpha = 2$  the Gaussian model (3.2)–(3.3), but it shows that in numerical practice when  $\alpha$  is near 2 the convergence is expected to be slow. For  $0 < \alpha < 2$  we obtain the generating functions and the transition probabilities as follow. Putting  $\rho(z) = [(1 - z)^{\alpha-1} - 1]$  for  $\alpha \neq 1$ , and introducing the scaling parameter  $\mu$ , we have

$$\tilde{p}(z) = \begin{cases} 1 - \frac{\mu}{2 \cos(\alpha\pi/2)} [(1 - z^{-1})\rho(z) + (1 - z)\rho(z^{-1})], & \alpha \neq 1, \\ 1 - \frac{\mu}{\pi} [(1 - z^{-1})\log(1 - z) + (1 - z)\log(1 - z^{-1})], & \alpha = 1. \end{cases} \quad (3.8)$$

For  $0 < \alpha < 2$ ,  $\alpha \neq 1$ , we have  $0 < \mu \leq \cos(\alpha\pi/2)/(1 - \alpha)$  and

$$\begin{cases} p_0 = 1 - \frac{\mu(1 - \alpha)}{\cos(\alpha\pi/2)}, \\ p_{\pm k} = (-1)^k \frac{\mu}{2 \cos(\alpha\pi/2)} \binom{\alpha}{k + 1} \end{cases} \quad \text{for } k = 1, 2, 3, \dots, \quad (3.9)$$

for  $\alpha = 1$  we have  $0 < \mu \leq \pi/2$  and

$$p_0 = 1 - \frac{2\mu}{\pi}, \quad p_{\pm k} = \frac{\mu}{\pi} \frac{1}{k(k + 1)} \quad \text{for } k = 1, 2, 3, 4, \dots \quad (3.10)$$

In both models (RW1), (RW2), the generating function is expressed in terms of elementary functions and the transition coefficients, for  $0 < \alpha < 2$ , exhibit an asymptotic behaviour as  $|k| \rightarrow \infty$  consistent with that of the power-law tails of the stable densities as  $|x| \rightarrow \infty$ , see, e.g. [1]. Indeed we obtain

$$p_k \sim \mu \Gamma(\alpha + 1) \frac{\sin(\pi\alpha/2)}{\pi} |k|^{-(\alpha+1)} \quad \text{as } |k| \rightarrow \infty, \quad 0 < \alpha < 2; \quad (3.11)$$

$$p_\alpha(x; 0) \sim \Gamma(\alpha + 1) \frac{\sin(\pi\alpha/2)}{\pi} |x|^{-(\alpha+1)} \quad \text{as } |x| \rightarrow \infty, \quad 0 < \alpha < 2. \quad (3.12)$$

The asymptotic behaviour of the stable densities is the starting point for the random walk model (RW3) in that, following Gillis and Weiss [14], we require that all  $p_k$  for  $k \neq 0$  are proportional to  $|k|^{-(\alpha+1)}$ . However, the simplicity of the starting point leads to difficulties for treating this model since the generating function is no longer elementary, a fact that makes the convergence proof a really hard affair. We need to recall the following special functions

$$\zeta(\beta) = \sum_{k=1}^{\infty} k^{-\beta}, \quad \beta > 1, \quad \Phi(z, \beta) = \sum_{k=1}^{\infty} \frac{z^k}{k^\beta}, \quad |z| < 1, \quad \beta \in \mathbf{R}, \quad (3.13)$$

respectively, known as the Riemann *zeta* function and the *polylogarithmic* function. This model has been extensively discussed in [9] (see also [14]), where the following generating function is derived:

$$\tilde{p}(z) = 1 - 2\lambda\zeta(\alpha + 1) + \lambda[\Phi(z, \alpha + 1) + \Phi(z^{-1}, \alpha + 1)]. \quad (3.14)$$

Here  $\beta = \alpha + 1 > 1$  so  $\Phi(z, \beta)$  is by its power series also defined on the periphery  $|z| = 1$  of the unit circle. Then we get a pure power-law random walk with

$$p_0 = 1 - 2\lambda\zeta(\alpha + 1), \quad p_k = \lambda|k|^{-(\alpha+1)} \quad \text{for } k \neq 0, \quad (3.15)$$

where  $\lambda$ , subject to the condition  $0 < \lambda \leq 1/[2\zeta(\alpha + 1)]$ , is to be determined. To ensure the convergence to the corresponding continuous process the following scaling relation must hold:

$$\tau = \sigma(h) = \begin{cases} \frac{\lambda\pi}{\Gamma(\alpha + 1)\sin(\alpha\pi/2)} h^\alpha & \text{if } 0 < \alpha < 2, \\ \lambda h^2 |\log h| & \text{if } \alpha = 2. \end{cases} \quad (3.16)$$

It is interesting to note that in the case  $\alpha = 2$  the classical random walk model (3.2)–(3.3) is no longer recovered, since now arbitrarily large jumps occur (with a probability decaying as  $k^{-3}$ ) as in the Lévy flights. Nevertheless, through the scaling relation (3.16), this random walk converges to the continuous Gaussian process. Thus this model gives us the opportunity to verify that a random walk with infinite variance steps may (slowly) converge to the Gaussian, in agreement with a general theorem on the domain of attraction of the normal law, see, e.g. [15].

#### 4. Numerical results

In general the random walk models are not only valuable from the conceptual point of view for visualizing what the diffusion means but also for numerical calculations, either as Monte Carlo simulation of particle paths in a diffusion process or as discrete imitation of the process in form of redistribution (from one time level to the next) of clumps of an extensive quantity (across the spatial grid points). Our models can be used in at least three different ways: (a) as *finite difference schemes* for approximate calculation of symmetric stable densities; (b) for producing *sample paths* of individual particles performing the random walk; (c) for producing *histograms* of the approximate

realization of the densities  $g_\alpha$  by simulating many individual paths with the same number of time steps and making statistics of the final positions of the particles.

For numerical simulations of stable random variables different algorithms have been provided by a number of specialists, including Chambers et al. [16], Bartels [17], Mantegna [18], Janicki and Weron [19], Samorodnitsky and Taqqu [20]. Our present approach for treating Lévy statistics has been carried out independent of all the above references but uniquely based on the random walk models presented here, so, as far as we know, our results would be in the great part original.

Having preliminarily checked a sufficient level of accuracy for our finite difference schemes with the existing tables of stable densities, here we present some results on the simulation of the sample paths and histograms corresponding to some typical values of the index of stability, namely  $\alpha = 1, 1.5, 2$ , in Figs. 1, 2, 3, respectively. In practice, in our numerical studies truncation is required in two ways. It is impossible to simulate all infinitely many discrete probabilities, so the size of possible jumps must be limited to a maximal possible jump length. The other truncation is required if a priori one wants a definite region of space to be considered in which the walk takes place. Then, particles leaving this space have been ignored. Our simulations, based on one million of realizations, have been carried out in the interval  $|x| \leq 4$ . All the histograms refer to stable densities at  $t = 1$  for  $|x| \leq 3$ , the space interval being reduced to avoid the border effects. The sample paths are plotted against the time steps, up to 1200 for  $\alpha = 1$  and up to 2000 for  $\alpha = 1.5, 2$ , so they refer to different final times, namely  $t = 1.87, 1.33, 0.8$ , respectively. The transition probabilities have been chosen from our random walk models as follows:  $\alpha = 1$  from (RW2), see (3.10), with scaling parameter  $\mu = \pi/4$ ;  $\alpha = 1.5$ , from (RW1), see (3.5b), with  $\mu = (2/3)\cos(3\pi/4)$ ; for  $\alpha = 2$ , from the standard model, see (3.2), with  $\mu = 1/4$ . The cases  $\alpha = 1$  (Cauchy process) and  $\alpha = 2$  (normal process) have been considered for a possible comparison with the standard

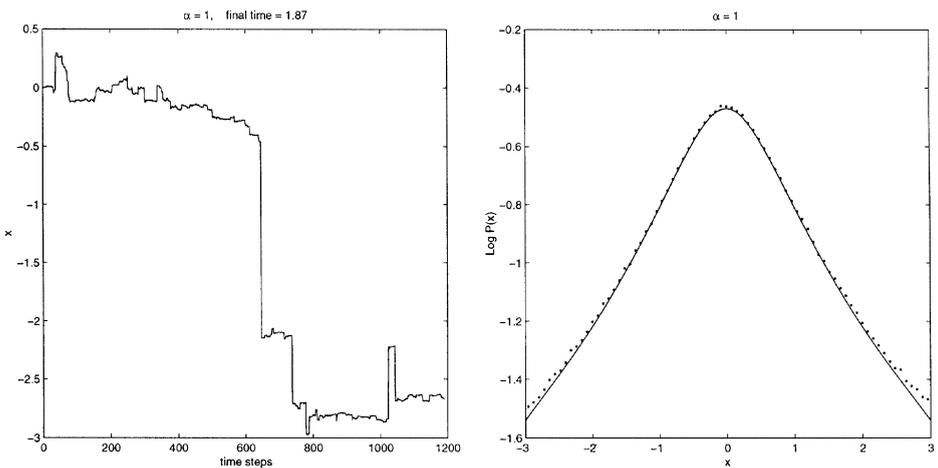


Fig. 1. Sample path (left) and histogram (right) for  $\alpha = 1$  (Cauchy).

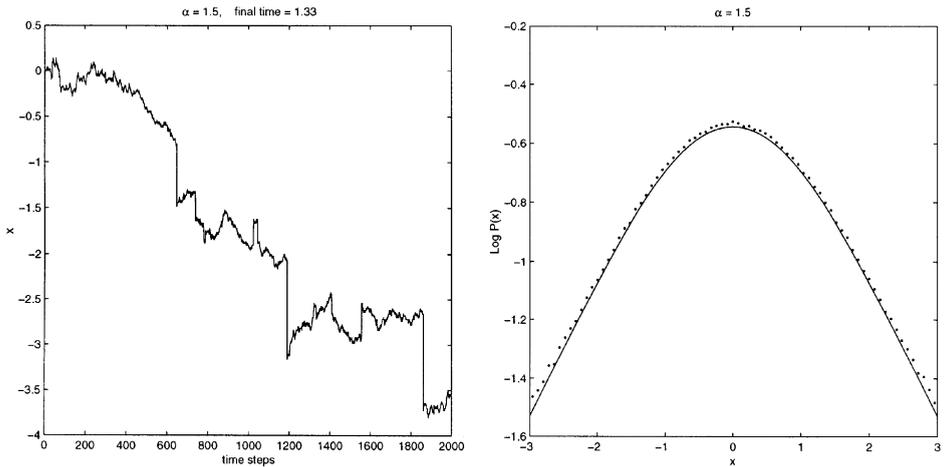


Fig. 2. Sample path (left) and histogram (right) for  $\alpha = 1.5$ .

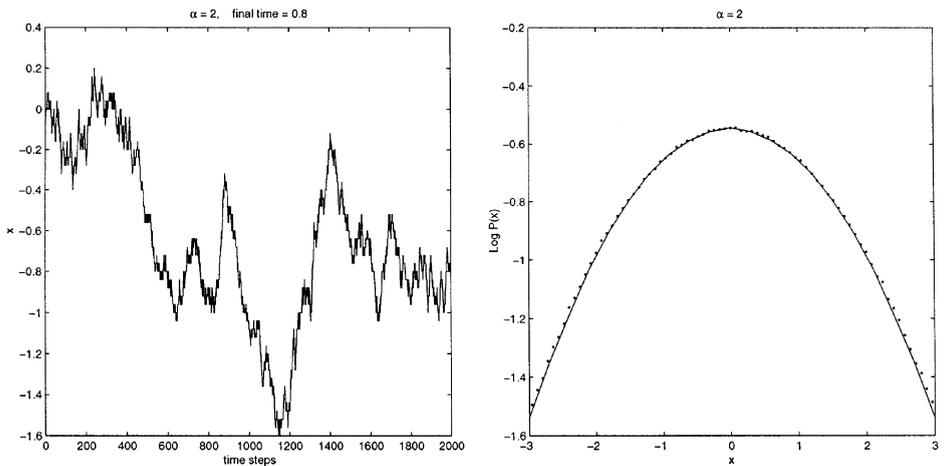


Fig. 3. Sample path (left) and histogram (right) for  $\alpha = 2$  (Gauss).

and accurate algorithms existing in the literature, whereas  $\alpha = 1.5$  has been chosen in view of possible applications in econophysics where usually the index of stability ranges from 1.4 to 1.7, see, e.g. [10,11].

## 5. Conclusions

For the simulation of Markovian processes characterized by symmetric Lévy probability densities evolving in time we have presented three different random walk models, discrete in space and time, by giving their respective transition probabilities. We have

indicated how, by use of generating functions, convergence for properly scaled transition to vanishing steps of space and time can be analysed, and we have hinted at peculiarities occurring at the particular stability indices  $\alpha = 1$  (the Cauchy process) and  $\alpha = 2$  (the Gauss process). We have displayed preliminary results of a few numerical case studies concerning sample paths and histograms to check the efficiency of our algorithms. From the sample paths one can recognize the “wild” character of the Lévy flights with respect to the “tame” character of the Brownian motion.

We expect that our arguments can be relevant in different fields of physics including the emerging one of econophysics, where stable distributions are becoming more common. In statistical physics the stable distributions play a key role in the (wonderful) world of random walks constructed by the late Montroll and continued through his school, see, e.g. [21–23]. Also Tsallis and his associates have recognized the key role of stable distributions with respect to a generalized theory of thermostatics, see, e.g. [24], and references therein. Here, we have (only briefly) pointed out the relation between Lévy statistics and space-fractional diffusion equations. However, stable distributions turn out to be related also with time-fractional diffusion equations, see, e.g. [25]. Furthermore, this topic is relevant for fractal phenomena, where differential equations of fractional order are usually adopted to describe their evolution, see, e.g. [26–28], and reference herein.

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