

ON SOME PROPERTIES OF THE MITTAG-LEFFLER
FUNCTION $E_\alpha(-t^\alpha)$, COMPLETELY MONOTONE FOR $t > 0$
WITH $0 < \alpha < 1$

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ABSTRACT. We analyse some peculiar properties of the function of the Mittag-Leffler (M-L) type, $e_\alpha(t) := E_\alpha(-t^\alpha)$ for $0 < \alpha < 1$ and $t > 0$, which is known to be completely monotone (CM) with a non-negative spectrum of frequencies and times, suitable to model fractional relaxation processes. We first note that (surprisingly) these two spectra coincide so providing a universal scaling property of this function, not well pointed out in the literature. Furthermore, we consider the problem of approximating our M-L function with simpler CM functions for small and large times. We provide two different sets of elementary CM functions that are asymptotically equivalent to $e_\alpha(t)$ as $t \rightarrow 0$ and $t \rightarrow +\infty$. The first set is given by the stretched exponential for small times and the power law for large times, following a standard approach. For the second set we chose two rational CM functions in t^α , obtained as the Padè Approximants (PA) $[0/1]$ to the convergent series in positive powers (as $t \rightarrow 0$) and to the asymptotic series in negative powers (as $t \rightarrow \infty$), respectively. From numerical computations we are allowed to the conjecture that the second set provides upper and lower bounds to the Mittag-Leffler function.

1. Introduction. Since a few decades the special transcendental function known as Mittag-Leffler function has attracted an increasing attention of researchers because of its key role in treating problems related to integral and differential equations of fractional order.

Since its introduction by the Swedish mathematician Mittag-Leffler at the beginning of the last century up to the 1990's, this function was seldom considered by mathematicians and applied scientists.

Before the 1990's, from a mathematical point of view, we recall the 1930 paper by Hille and Tamarkin [18] on the solutions of the Abel integral equation of the second kind, and the books by Davis [8], Sansone & Gerretsen [40], Dzherbashyan [10] (unfortunately in Russian), and finally Samko et al. [38]. Particular mention would be for the 1955 Handbook of High Transcendental Functions of the Bateman project [11], where this function was treated in the chapter devoted to miscellaneous functions. For former applications we recall an interesting note by Davis [8] reporting a previous research by Dr. Kenneth S. Cole in connection with nerve conduction, and the papers by Cole & Cole [7], Gross [16] and Caputo & Mainardi

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[5, 6], where the Mittag-Leffler function was adopted to represent the responses in dielectric and viscoelastic media.

In the 1960's the Mittag-Leffler function started to exit from the realm of miscellaneous functions because it was considered as a special case of the general class of Fox H functions, that can exhibit an arbitrary number of parameters in their integral Mellin-Barnes representation, see e.g. the books by Kiryakova [23], Kilbas and Saigo [21], Marichev [30], Mathai & Saxena [32], Mathai et al. [33], Srivastava et al. [43].

However, in our opinion, this classification in a too general framework has, to some extent, obscured the relevance and the applicability of this function in applied science. In fact most mathematical models are based on a small number of parameters, say 1 or 2 or 3, so that a general theory may be confusing whereas the adoption of a generalized Mittag-Leffler function with 2 or 3 indices may be sufficient, see e.g. Beghin & Orsingher [3], Capelas et al. [4], Sandev et al. [39], Tomovski et al. [46]. Multi-index Mittag-Leffler functions have been introduced as well, see e.g. Kilbas et al. [19], Kilbas & Saigo [20], Kiryakova [24], Kiryakova & Luchko [25], but their extensive use has not yet been pointed out in applied sciences up to now.

Nowadays it is well recognized that the Mittag-Leffler function plays a fundamental role in Fractional Calculus even if with a single parameter (as originally introduced by Mittag-Leffler) just to be worth of being referred to as the *Queen Function of Fractional Calculus*, see Mainardi & Gorenflo [29]. On this respect we just point out some reviews and treatises on Fractional Calculus (in order of publication time): Gorenflo & Mainardi [15], Podlubny [35], Hilfer [17], Kilbas et al. [22], Magin [27], Mathai & Haubold [31], Mainardi [28], Diethelm [9], Tarasov [45], Klafter et al. [26], Baleanu et al. [2], Uchaikin [47].

2. The Mittag-Leffler function in fractional relaxation processes. The Mittag-Leffler function is defined by the following power series, convergent in the whole complex plane,

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \quad (2.1)$$

We recognize that it is an entire function providing a simple generalization of the exponential function to which it reduces for $\alpha = 1$. We also note that for the convergence of the power series in (2.1) the parameter α may be complex provided that $\Re(\alpha) > 0$. The most interesting properties of the Mittag-Leffler function are associated with its asymptotic expansions as $z \rightarrow \infty$ in various sectors of the complex plane. For detailed asymptotic analysis, which includes the smooth transition across the Stokes lines, the interested reader is referred to Wong and Zhao [48].

In this paper we limit ourselves to the Mittag-Leffler function of order $\alpha \in (0, 1)$ on the negative real semi-axis where is known to be completely monotone (CM) due a classical result by Pollard [37], see also Feller [12].

Let us recall that a function $\phi(t)$ with $t \in \mathbb{R}^+$ is called a completely monotone (CM) function if it is non-negative, of class C^{∞} , and $(-1)^n \phi^{(n)}(t) \geq 0$ for all $n \in \mathbb{N}$. Then a function $\psi(t)$ with $t \in \mathbb{R}^+$ is called a Bernstein function if it is non-negative, of class C^{∞} , with a CM first derivative. These functions play fundamental roles in linear hereditary mechanics to represent relaxation and creep processes, see e.g. Mainardi [28]. For mathematical details we refer the interested reader to the survey paper by Miller and Samko [34] and to the recent book by Schilling et al. [41].

In particular we are interested to the function

$$e_\alpha(t) := E_\alpha(-t^\alpha) = \sum_{n=0}^\infty (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)}, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (2.2)$$

that provides the solution to the fractional relaxation equation, see Gorenflo and Mainardi [15], Mainardi and Gorenflo [29], Mainardi [28].

For readers' convenience let us briefly outline the topic concerning the generalization via fractional calculus of the first-order differential equation governing the phenomenon of (exponential) relaxation. Recalling (in non-dimensional units) the initial value problem

$$\frac{du}{dt} = -u(t), \quad t \geq 0, \quad \text{with } u(0^+) = 1 \quad (2.3)$$

whose solution is

$$u(t) = \exp(-t), \quad (2.4)$$

the following two alternatives with $\alpha \in (0, 1)$ are offered in the literature:

$$\frac{du}{dt} = -D_t^{1-\alpha} u(t), \quad t \geq 0, \quad \text{with } u(0^+) = 1, \quad (2.5a)$$

$${}_0^*D_t^\alpha u(t) = -u(t), \quad t \geq 0, \quad \text{with } u(0^+) = 1. \quad (2.5b)$$

where $D_t^{1-\alpha}$ and ${}_0^*D_t^\alpha$ denote the fractional derivative of order $1-\alpha$ in the Riemann-Liouville sense and the fractional derivative of order α in the Caputo sense, respectively.

For a generic order $\mu \in (0, 1)$ and for a sufficiently well-behaved function $f(t)$ with $t \in \mathbb{R}^+$ the above derivatives are defined as follows, see e.g. Gorenflo and Mainardi [15], Podlubny [35],

$$D_t^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \left[\int_0^t \frac{f(\tau)}{(t-\tau)^\mu} d\tau \right], \quad (2.6a)$$

$${}_0^*D_t^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\mu} d\tau. \quad (2.6b)$$

Between the two derivatives we have the relationship

$${}_0^*D_t^\mu f(t) = D_t^\mu f(t) - f(0^+) \frac{t^{-\mu}}{\Gamma(1-\mu)} = D_t^\mu [f(t) - f(0^+)]. \quad (2.7)$$

Both derivatives in the limit $\mu \rightarrow 1^-$ reduce to the standard first derivative but for $\mu \rightarrow 0^+$ we have

$$D_t^\mu f(t) \rightarrow f(t), \quad {}_0^*D_t^\mu f(t) = f(t) - f(0^+), \quad \mu \rightarrow 0^+. \quad (2.8)$$

In analogy to the standard problem (2.3), we solve the problems (2.5a) and (2.5b) with the Laplace transform technique, using the rules pertinent to the corresponding fractional derivatives. The problems (a) and (b) are equivalent since the Laplace transform of the solution in both cases comes out as

$$\tilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + 1}, \quad (2.9)$$

that yields our function

$$u(t) = e_\alpha(t) := E_\alpha(-t^\alpha). \quad (2.10)$$

The Laplace transform pair

$$e_\alpha(t) \div \frac{s^{\alpha-1}}{s^\alpha + 1}, \quad \alpha > 0, \quad (2.11)$$

can be proved by transforming term by term the power series representation of $e_\alpha(t)$ in the R.H.S of (2.2). Furthermore, by anti-transforming the R.H.S of (2.11) by using the complex Bromwich formula, and taking into account for $0 < \alpha < 1$ the contribution from branch cut on the negative real semi-axis (the denominator $s^\alpha + 1$ does nowhere vanish in the cut plane $-\pi \leq \arg s \leq \pi$), we get, see also Gorenflo and Mainardi [15],

$$e_\alpha(t) = \int_0^\infty e^{-rt} K_\alpha(r) dr, \tag{2.12}$$

where

$$K_\alpha(r) = \mp \frac{1}{\pi} \operatorname{Im} \left\{ \frac{s^{\alpha-1}}{s^\alpha + 1} \Big|_{s=r e^{\pm i\pi}} \right\} = \frac{1}{\pi} \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1} \geq 0. \tag{2.13}$$

Since $K_\alpha(r)$ is non-negative for all r in the integral, the above formula proves that $e_\alpha(t)$ is CM function in view of the Bernstein theorem. This theorem provides a necessary and sufficient condition for a CM function as a real Laplace transform of a non-negative measure.

However, the CM property of $e_\alpha(t)$ can also be seen as a consequence of the result by Pollard because the transformation $x = t^\alpha$ is a Bernstein function for $\alpha \in (0, 1)$. In fact it is known that a CM function can be obtained by composing a CM with a Bernstein function based on the following theorem: *Let $\phi(t)$ be a CM function and let $\psi(t)$ be a Bernstein function, then $\phi[\psi(t)]$ is a CM function.*

As a matter of fact, $K_\alpha(r)$ provides an interesting spectral representation of $e_\alpha(t)$ in frequencies. With the change of variable $\tau = 1/r$ we get the corresponding spectral representation in relaxation times, namely

$$e_\alpha(t) = \int_0^\infty e^{-t/\tau} H_\alpha(\tau) d\tau, \quad H_\alpha(\tau) = \tau^{-2} K_\alpha(1/\tau), \tag{2.14}$$

that can be interpreted as a continuous distributions of elementary (i.e. exponential) relaxation processes. As a consequence we get the identity between the two spectral distributions, that is

$$H_\alpha(\tau) = \frac{1}{\pi} \frac{\tau^{\alpha-1} \sin(\alpha\pi)}{\tau^{2\alpha} + 2\tau^\alpha \cos(\alpha\pi) + 1}, \tag{2.15}$$

a surprising fact pointed out in Linear Viscoelasticity by the author in his book [28]. This kind of universal/scaling property seems a peculiar one for our Mittag-Leffler function $e_\alpha(t)$. In Fig 1 we show $K_\alpha(r)$ for some values of the parameter α . Of course for $\alpha = 1$ the Mittag-Leffler function reduces to the exponential function $\exp(-t)$ and the corresponding spectral distribution is the Dirac delta generalized function centred at $r = 1$, namely $\delta(r - 1)$.

3. Asymptotic approximations to the Mittag-Leffler function. In Fig 2 we show some plots of $e_\alpha(t)$ for some values of the parameter α . It is worth to note the different rates of decay of $e_\alpha(t)$ for small and large times. In fact the decay is very fast as $t \rightarrow 0^+$ and very slow as $t \rightarrow +\infty$.

3.1. The two common asymptotic approximations. It is common to point out that the function $e_\alpha(t)$ matches for $t \rightarrow 0^+$ with a stretched exponential with an infinite negative derivative, whereas as $t \rightarrow \infty$ with a negative power law. The short time approximation is derived from the convergent power series representation (2.2). In fact,

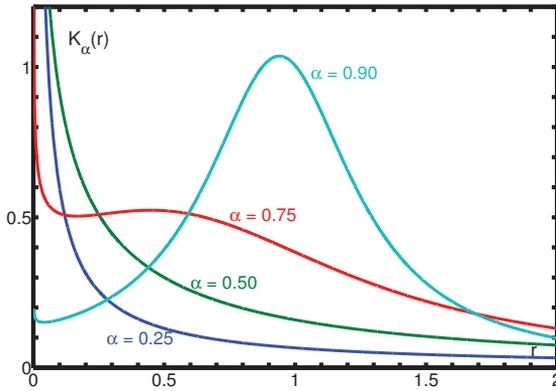


Fig.1 Plots of the spectral function $K_\alpha(r)$ for $\alpha = 0.25, 0.50, 0.75, 0.90$ in the frequency range $0 \leq r \leq 2$.

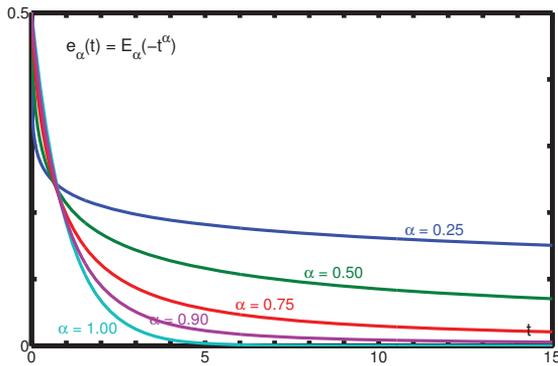


Fig.2 Plots of the Mittag-Leffler function $e_\alpha(t)$ for $\alpha = 0.25, 0.50, 0.75, 0.90, 1.$ in the time range $0 \leq t \leq 15$

$$e_\alpha(t) = 1 - \frac{t^\alpha}{\Gamma(1 + \alpha)} + \dots \sim \exp \left[-\frac{t^\alpha}{\Gamma(1 + \alpha)} \right], \quad t \rightarrow 0. \tag{3.1}$$

The long time approximation is derived from the asymptotic power series representation of $e_\alpha(t)$ that turns out to be, see Erdélyi (1955),

$$e_\alpha(t) \sim \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{-\alpha n}}{\Gamma(1 - \alpha n)}, \quad t \rightarrow \infty, \tag{3.2}$$

so that, at the first order,

$$e_\alpha(t) \sim \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \quad t \rightarrow \infty. \tag{3.3}$$

As a consequence the function $e_\alpha(t)$ interpolates for intermediate time t between the stretched exponential and the negative power law. The stretched exponential models the very fast decay for small time t , whereas the asymptotic power law is due to the very slow decay for large time t . In fact, we have the two commonly

stated asymptotic representations:

$$e_\alpha(t) \sim \begin{cases} e_\alpha^0(t) := \exp\left[-\frac{t^\alpha}{\Gamma(1+\alpha)}\right], & t \rightarrow 0; \\ e_\alpha^\infty(t) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\sin(\alpha\pi)}{\pi} \frac{\Gamma(\alpha)}{t^\alpha}, & t \rightarrow \infty. \end{cases} \quad (3.4)$$

The stretched exponential replaces the rapidly decreasing expression $1 - t^\alpha/\Gamma(1 + \alpha)$ from (3.1). Of course, for sufficiently small and for sufficiently large values of t we have the inequality

$$e_\alpha^0(t) \leq e_\alpha^\infty(t), \quad 0 < \alpha < 1. \quad (3.5)$$

In Figs 3-7 LEFT we compare for $\alpha = 0.25, 0.5, 0.75, 0.90, 0.99$ in logarithmic scales the function $e_\alpha(t)$ (continuous line) and its asymptotic representations, the stretched exponential $e_\alpha^0(t)$ valid for $t \rightarrow 0$ (dashed line) and the power law $e_\alpha^\infty(t)$ valid for $t \rightarrow \infty$ (dotted line). We have chosen the time range $10^{-5} \leq t \leq 10^{+5}$. In the RIGHT we have shown the plots of the relative errors (in absolute values)

$$\frac{|e_\alpha^0(t) - e_\alpha(t)|}{e_\alpha(t)}, \quad \frac{|e_\alpha^\infty(t) - e_\alpha(t)|}{e_\alpha(t)}, \quad (3.6)$$

pointing out a continuous line at an error 1% under which the approximations can be considered reliable.

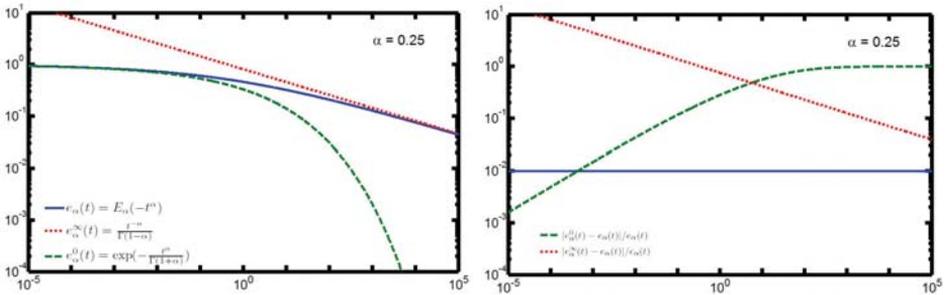


Fig.3 Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.25$.

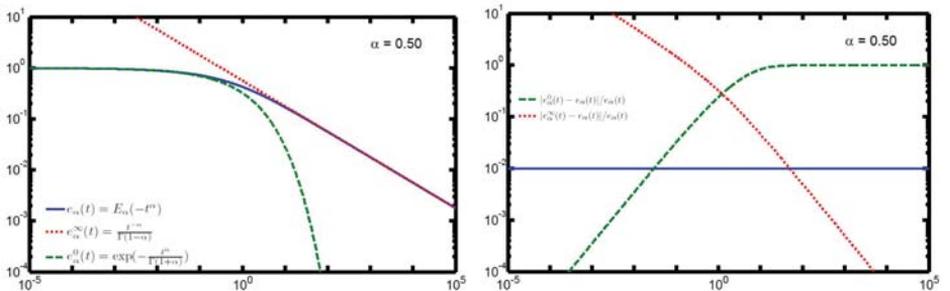


Fig.4 Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.50$.

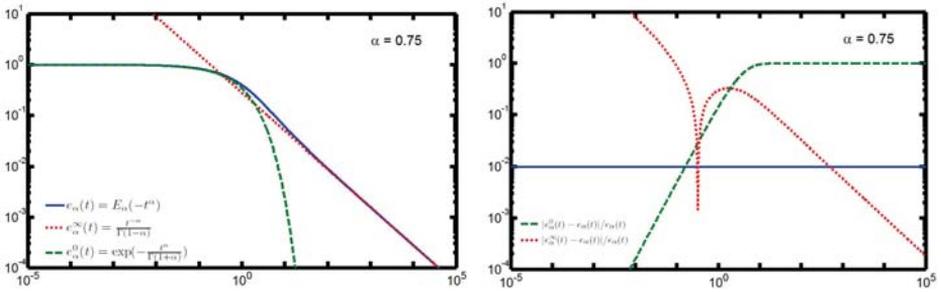


Fig.5 Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^5$ for $\alpha = 0.75$.

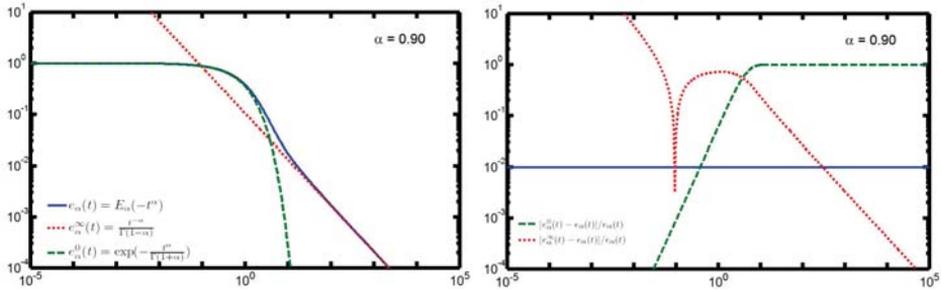


Fig.6 Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^5$ for $\alpha = 0.90$.

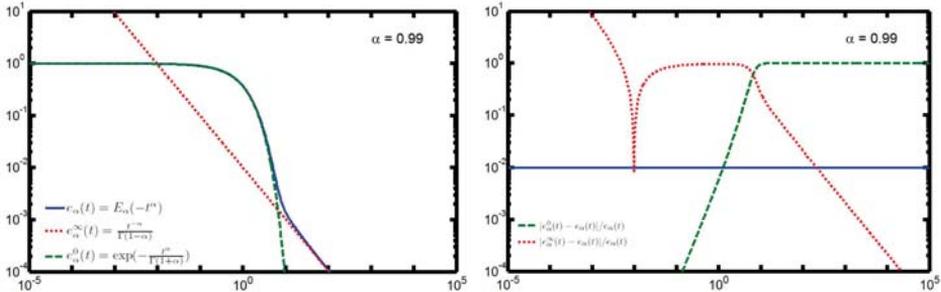


Fig.7 Approximations $e_\alpha^0(t)$ (dashed line) and $e_\alpha^\infty(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^5$ for $\alpha = 0.99$.

We note from Figs 3-7 that, whereas the plots of $e_\alpha^0(t)$ remain always under the corresponding ones of $e_\alpha(t)$, the plots of $e_\alpha^\infty(t)$ start above those of $e_\alpha(t)$ but, at a certain point, an intersection may occur so changing the sign of the relative errors. The steep cusps occurring to the left of $t = 10^0$ in Figs 5, 6, and 7 indicates that there the relative error is falling down to zero.

3.2. The two rational asymptotic approximations. We now propose a new set of CM functions approximating $e_\alpha(t)$: $\{f_\alpha(t), g_\alpha(t)\}$, alternative to $\{e_\alpha^0(t), e_\alpha^\infty(t)\}$, obtained as the first Padè approximants $[0/1]$ to the power series in t^α (2.2) and (3.2), respectively. For more details on the theory of Padè Approximants we refer

e.g. to Baker [1]. We thus obtain the following rational functions in t^α :

$$f_\alpha(t) := \frac{1}{1 + \frac{t^\alpha}{\Gamma(1+\alpha)}} \sim 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \sim e_\alpha(t), \quad t \rightarrow 0, \tag{3.7}$$

$$g_\alpha(t) := \frac{1}{1 + t^\alpha \Gamma(1-\alpha)} \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \sim e_\alpha(t), \quad t \rightarrow \infty. \tag{3.8}$$

Now we prove the inequality

$$g_\alpha(t) \leq f_\alpha(t), \quad t \geq 0, \quad 0 < \alpha < 1, \tag{3.9}$$

as a straightforward consequence of the reflection formula of the gamma function. In fact, recalling the definitions (3.7)-(3.8) we have for $\forall t \geq 0$ and $\alpha \in (0, 1)$:

$$g_\alpha(t) \leq f_\alpha(t) \iff \Gamma(1-\alpha) \geq \frac{1}{\Gamma(1+\alpha)} \iff \Gamma(1-\alpha)\Gamma(1+\alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)} \geq 1.$$

In Figs 8-12 LEFT we compare for $\alpha = 0.25, 0.5, 0.75, 0.90, 0.99$ in logarithmic scales the function $e_\alpha(t)$ (continuous line) and its rational asymptotic representations, $f_\alpha(t)$ valid for $t \rightarrow 0$ (dashed line) and $g_\alpha(t)$ valid for $t \rightarrow \infty$ (dotted line). We have chosen the time range $10^{-5} \leq t \leq 10^{+5}$. In the RIGHT we have shown the plots of the relative errors (no longer in absolute values)

$$\frac{f_\alpha(t) - e_\alpha(t)}{e_\alpha(t)}, \quad \frac{e_\alpha(t) - g_\alpha(t)}{e_\alpha(t)}, \tag{3.10}$$

pointing out a continuous line at an error 1% under which the approximation can be considered reliable.

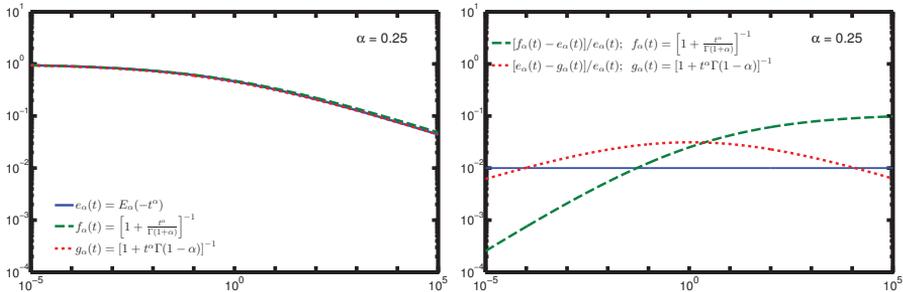


Fig.8 Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.25$.

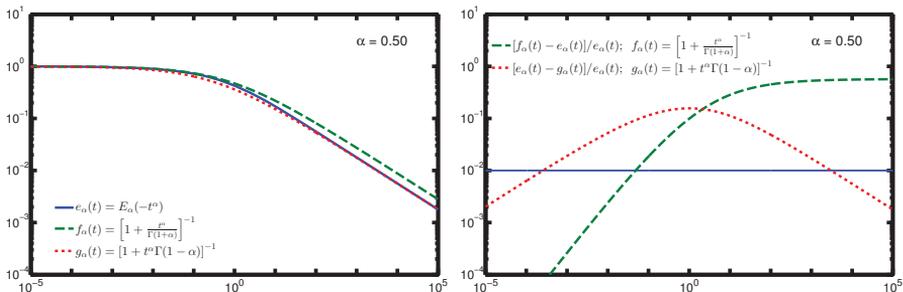


Fig.9 Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.50$.

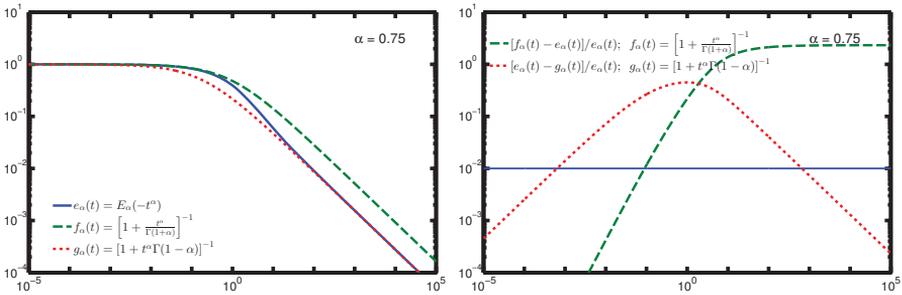


Fig.10 Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.75$.

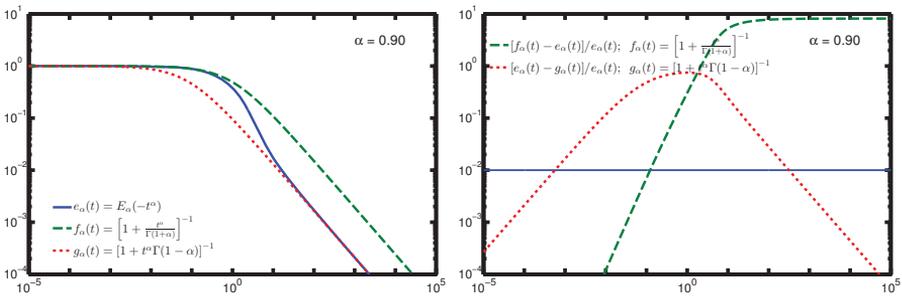


Fig.11 Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.90$.

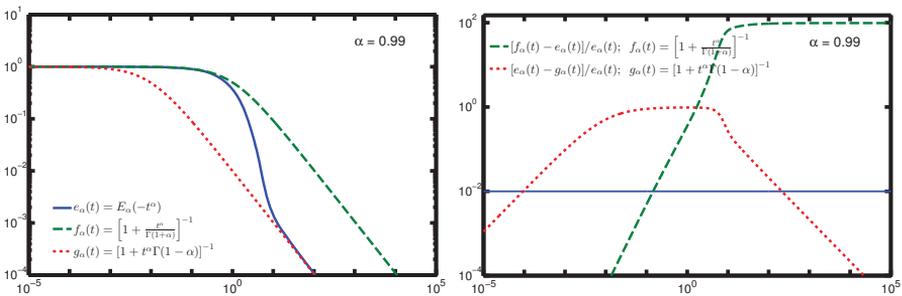


Fig.12 Approximations $f_\alpha(t)$ (dashed line) and $g_\alpha(t)$ (dotted line) to $e_\alpha(t)$ (LEFT) and the corresponding relative errors (RIGHT) in $10^{-5} \leq t \leq 10^{+5}$ for $\alpha = 0.99$.

As a matter of fact, from the plots in Figs 8-12, we recognize, for the time range and for $\alpha \in (0, 1)$ considered by us, the relevant inequality

$$g_\alpha(t) \leq e_\alpha(t) \leq f_\alpha(t), \tag{3.11}$$

that is $g_\alpha(t)$ and $f_\alpha(t)$ provide lower and upper bounds to our Mittag-Leffler function $e_\alpha(t)$. This of course can be seen as a conjecture that we leave as an open problem to be proved (or disproved) by specialists of CM functions.

We also see that whereas the short time approximation $f_\alpha(t)$ turns out to be good only for small times, the long time approximation $g_\alpha(t)$ is good (surprisingly) also for short times, falling down only in an intermediate time range. In fact from the RIGHT of Figs 8-12 we can estimate the ranges of validity when the relative error is less than 1%. We could further discuss on them.

Concerning Padé Approximants (PA) for the Mittag-Leffler functions $E_\alpha(-x)$ for $x > 0$ and $0 < \alpha < 1$ we like to cite Freed et al. [13] for an appendix devoted to the table of PA, Starovotov and Starovotova [44] for theorems on the best uniform rational approximations. Quite recently Zeng & Chen [49] have found a global Padé approximation with degree 2 of the generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ with $x > 0$, $0 < \alpha < 1$ and $\beta \geq \alpha$. In this interesting pre-print the uniform approximation can account for both the Taylor series for small arguments and asymptotic series for large arguments, but no upper and lower bounds are obtained as in our case.

We point out the fact that our first PA's [0/1] are constructed from two different series (in positive and negative powers of t^α) both resulting CM functions whereas the successive PA's of higher order can exhibit oscillations for $t > 0$.

4. Conclusions. We have discussed some noteworthy properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ with $0 < \alpha < 1$ in the range $t > 0$.

Being a completely monotone (CM) function, because of the Bernstein theorem this function admits non-negative frequency and time spectra. We have pointed out that these two spectra are equal, so providing a universal scaling property.

Furthermore, in view of a numerical approximation we have compared two different sets of approximating CM functions, asymptotically equivalent to $E_\alpha(-t^\alpha)$ for small and large times: the former commonly used in the literature, the latter probably new. The last set is constituted by two simple rational functions that provide upper and lower bounds, at least in our numerical examples on a large time range. We are allowed to conjecture that this bounding property is always valid for any $t > 0$ and for any $\alpha \in (0, 1)$: an open problem offered to specialists of CM functions. Last but not the least, we note that our rational function approximating $E_\alpha(-t^\alpha)$ for large times provides (surprisingly) a good numerical approximation also for short times, so failing only in an intermediate range.

Final remark. Since the appearance of the first version (May 2013), this paper was submitted to the attention of some colleagues and to any interested reader of arXiv in order to get the proof of our conjecture. Only recently (October 2013) a proof was provided by Thomas Simon (University of Lille, France) based on probability arguments, see [42]. We note, however, that in July 2013 Renato Spigler (University of Rome 3, Italy) proved, in part, the conjecture made in eq. (3.11), for $x := t^\alpha$ in some right neighborhood of $x = 0$. This set is the most critical in establishing such conjecture, see Figs 8 – 12. See R. Spigler, 2013 SIAM Annual Meeting, July 8-12, 2013, San Diego, CA (USA), MS 119, minisymposium on “Special Functions: Applications and Numerical Aspects” – Part II of II, p. 100.

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REFERENCES

- [1] G. A. Baker, *Essentials of Padé Approximants*, Academic Press, New York, 1975.

- [2] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus. Models and Numerical Methods*, World Scientific, Singapore, 2012.
- [3] L. Beghin and E. Orsingher, *Poisson-type processes governed by fractional and higher-order recursive differential equations*, *Electron. J. Probab.*, **15** (2010), 684–709.
- [4] E. Capelas de Oliveira, F. Mainardi and J. Vaz Jr, *Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics*, *Eur. Phys. J., Special Topics*, **193** (2011) 161–171. [E-print arxiv.org/abs/1106.1761]
- [5] M. Caputo and F. Mainardi, *A new dissipation model based on memory mechanism*, *Pure and Appl. Geophys. (PAGEOPH)*, **91** (1971), 134–147. [Reprinted in *Fract. Calc. Appl. Anal.*, **10** (2007), 309–324.]
- [6] M. Caputo and F. Mainardi, *Linear models of dissipation in anelastic solids*, *Riv. Nuovo Cimento (Ser. II)*, **1** (1971), 161–198.
- [7] K. S. Cole and R. H. Cole, *Dispersion and absorption in dielectrics, II. Direct current characteristics*, *J. Chemical Physics*, **10** (1942), 98–105.
- [8] H. T. Davis, *The Theory of Linear Operators*, The Principia Press, Bloomington, Indiana, 1936.
- [9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Lecture Notes in Mathematics No 2004, Heidelberg, 2010.
- [10] M. M. Dzherbashyan, , *Integral Transforms and Representations of Functions in the Complex Plane*, Nauka, Moscow., 1966 [in Russian].
- [11] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. III. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [12] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, Second Edition, Wiley, New York, 1971.
- [13] A. Freed, K. Diethelm and Y. Luchko, *Fractional-order Viscoelasticity (FOV): Constitutive Development using the Fractional Calculus*, First Annual Report, NASA/TM-2002-211914, Glenn Research Center, 2002, pp. XIV – 121.
- [14] R. Gorenflo, J. Loutchko and Y. Luchko, *Computation of the Mittag-Leffler function and its derivatives*, *Fract. Calc. Appl. Anal.*, **5** (2002), 491–518.
- [15] R. Gorenflo and F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, in *Fractals and Fractional Calculus in Continuum Mechanics*, (eds. A. Carpinteri and F. Mainardi), Springer Verlag, Wien, 1997, pp. 223–276. [E-print arxiv.org/abs/0805.3823]
- [16] B. Gross, *On creep and relaxation*, *J. Appl. Phys.*, **18** (1947), 212–221.
- [17] R. Hilfer (editor), *Fractional Calculus, Applications in Physics*, World Scientific, Singapore, 2000.
- [18] E. Hille and J. D. Tamarkin, *On the theory of linear integral equations*, *Ann. Math.*, **31** (1930), 479–528.
- [19] A. A. Kilbas, A . A. Koroleva and S. V. Rogosin, *Multi-parametric Mittag-Leffler functions and their extensions*, *Fract. Calc. Appl. Anal.*, **16** (2013), 378–404.
- [20] A. A. Kilbas and M. Saigo, *On solution of integral equations of Abel-Volterra type*, *Differential and Integral Equations*, **8** (1995), 993–1011.
- [21] A. A. Kilbas and M. Saigo, *H-Transforms. Theory and Applications*, Chapman and Hall/CRC, Boca Raton, FL, 2004.
- [22] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [23] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow - New York, 1994.
- [24] V. Kiryakova, *The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus*, *Comp. Math. Appl.*, **59** (2010), 1885–1895.
- [25] V. Kiryakova and Y. Luchko, *The multi-index Mittag-Leffler functions and their applications for solving fractional order problems in applied analysis*, In *American Institute of Physics - Conf. Proc.*, **1301** (2010), 597–613.
- [26] J. Klafter, S. C. Lim and R. Metzler (Editors), *Fractional Dynamics, Recent Advances*, World Scientific, Singapore, 2012.
- [27] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, Connecticut, 2006.

- [28] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London and World Scientific, Singapore, 2010.
- [29] F. Mainardi and R. Gorenflo, Time-fractional derivatives in relaxation processes: A tutorial survey, *Fract. Calc. Appl. Anal.*, **10** (2007), 269–308.
- [30] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood, Chichester, 1983.
- [31] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.
- [32] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Wiley Eastern Ltd, New Delhi, 1978.
- [33] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Function: Theory and Applications*, Springer Verlag, New York, 2010.
- [34] K. S. Miller and S. G. Samko, Completely monotonic functions, *Integral Transforms and Special Functions*, **12** (2001), 389–402.
- [35] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [36] I. Podlubny, Mittag-Leffler function, Matlab-Code that calculates the Mittag-Leffler function with desired accuracy, Matlab File Exchange www.mathworks.com/matlabcentral/fileexchange, 2006.
- [37] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, *Bull. Amer. Math. Soc.*, **54** (1948), 1115–1116.
- [38] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam, 1993. [English translation and revised version from the Russian edition, Integrals and Derivatives of Fractional Order and Some of Their Applications Nauka i Tekhnika, Minsk, 1987]
- [39] T. Sandev, R. Metzler and Z. Tomovski, Velocity and displacement correlation functions for fractional generalized Langevin equations, *Fract. Calc. Appl. Anal.*, **15** (2012), 426–450.
- [40] G. Sansone and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable*, Vol. I. Holomorphic Functions, Nordhoff, Groningen, 1960.
- [41] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions. Theory and Applications*, 2-nd ed., De Gruyter, Berlin, 2012.
- [42] T. Simon, Comparing Fréchet and positive stable laws, *Electron. J. Probab.*, **19** (2014), 1–25. [E-print arXiv:1310.1888]
- [43] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.
- [44] A. P. Starovoitov and N. A. Starovoitova, Padè approximants of the Mittag-Leffler functions, *Sbornik Mathematics*, **198** (2007), 1011–1023.
- [45] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, 2010.
- [46] Z. Tomovski, R. Hilfer and H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Integral Transforms and Special Functions*, **21** (2010), 797–814.
- [47] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers*, Springer, Berlin, 2013.
- [48] R. Wong and Y.-Q. Zhao, Exponential asymptotics of the Mittag-Leffler function, *Constructive Approximation*, **18** (2002), 355–385.
- [49] C. Zeng and Y.-Q. Chen, Global Padè approximations for the generalized Mittag-Leffler function and its inverse, E-print arXiv:1310.5592 [math.CA] (2013), pp. 17.

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