LECTURE NOTES ON MATHEMATICAL PHYSICS
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THE EULERIAN FUNCTIONS

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1. THE GAMMA FUNCTION : $\Gamma(z)$

The Gamma function is the most widely used of all the special functions: it is usually discussed first since it appears in almost every integral or series representation of the other advanced mathematical functions. This function, denoted by $\Gamma(z)$, can be defined by

Euler's Integral Representation (1)

$$\boxed{\Gamma(z) = \int_0^\infty e^{-u}\, u^{z-1} du\,, \quad \mathcal{R}e(z) > 0\,.} \tag{G.1}$$

This representation is the most common for $\Gamma(z)$, even if it is valid only in the right half-plane of $\mathbb{C}$; later, the analytic continuation to the left half-plane will be considered to obtain its

Domain of Analyticity

$$\boxed{D_\Gamma = \mathbb{C} - \{0, -1, -2, \dots, \}\,.} \tag{G.2}$$

Using integration by parts, (G.1) shows that, at least for $\mathcal{R}e(z) > 0$, $\Gamma(z)$ satisfies the simple

Difference Equation (2)

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)\,,} \tag{G.3}$$

which can be iterated to yield

$$\boxed{\Gamma(z+n) = z\,(z+1)\,\dots\,(z+n-1)\,\Gamma(z)\,, \quad n \in \mathbb{N}\,.} \tag{G.4}$$

The recurrence formulas (G.3-4) can be extended to any $z \in D_\Gamma$. In particular, being $\Gamma(1) = 1$ we get for (non negative)

Integer Values

$$\boxed{\Gamma(n+1) = n!\,, \quad n = 0, 1, 2, \dots\,.} \tag{G.5}$$

As a consequence $\Gamma(z)$ can be used to define the

Complex Factorial Function

$$\boxed{z! \,:=\, \Gamma(z+1)\,.} \tag{G.6}$$

By the substitution $u = v^2$ in (G.1) we get the

Gaussian Integral Representation

$$\boxed{\Gamma(z) = 2\int_0^\infty e^{-v^2}\, v^{2z-1} dv\,, \quad \mathcal{R}e(z) > 0\,,} \tag{G.7}$$

which can be used to obtain $\Gamma(z)$ when z assumes positive semi-integer values, as follows.

Semi-Integer Values (3)

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{+\infty} e^{-v^2}\, dv = \sqrt{\pi} \approx 1.77245\,, \tag{G.8}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \int_{-\infty}^{+\infty} e^{-v^2}\, v^{2n}\, dv = \Gamma\left(\frac{1}{2}\right)\,\frac{(2n-1)!!}{2^n} = \sqrt{\pi}\,\frac{(2n)!}{2^{2n}\,n!}\,,\ \ n \in \mathbb{N}\,. \tag{G.9}$$

The formula (G.4), proven for $\mathcal{R}e(z) > 0$ using integration by parts in (G.1), can be used to obtain the *Domain of Analyticity* D_Γ by means of the so-called

Analytical Continuation by the Recurrence Formula (4)

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)\,(z+n-2)\ldots(z+1)\,z}\,. \tag{G.10}$$

In fact the numerator at the R.H.S of (G.10) is analytic for $\mathcal{R}e(z) > -n$; hence, the L.H.S. is analytic for $\mathcal{R}e(z) > -n$ except for simple poles at $z = 0, -1, \ldots, (-n+2), (-n+1)$. Since n can be arbitrarily large, we deduce that $\Gamma(z)$ is analytic in the entire complex plane except the points $z_n = -n$ $(n = 0, 1, \ldots)$, which turn out to be simple poles with residues $R_n = (-1)^n/n!$. The point at the infinity, being accumulation point of poles, is an essential non isolated singularity. Thus $\Gamma(z)$ is a transcendental *meromorphic* function.

The integration by parts in the basic representation (G.1) provides the

Analytical Continuation by the Cauchy-Saalschütz Representation

$$\Gamma(z) = \int_0^{\infty} u^{z-1}\left[\mathrm{e}^{-u} - 1 + u - \frac{u^2}{2!} + \ldots + (-1)^{n+1}\frac{u^n}{n!}\right] du\ , \tag{G.10$'$}$$

which holds for any integer $n \geq 0$ with $-(n+1) < \mathcal{R}e(z) < -n$. The representation can be understood by iterating the first step. In fact for $-1 < \mathcal{R}e(z) < 0$:

$$\int_0^{\infty} u^{z-1}\,\left[\mathrm{e}^{-u} - 1\right]\,du = \frac{1}{z}\int_0^{\infty} u^z\,\mathrm{e}^{-u}\,du = \frac{1}{z}\Gamma(z+1) = \Gamma(z)\,.$$

Finally another instructive manner to obtain the *Domain of Analyticity* is to use the so-called

Analytical Continuation by the Mixed Representation (5)

$$\Gamma(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-u}\,u^{z-1}\,du\,, \quad z \in D_\Gamma\,. \tag{G.11}$$

This representation can be obtained splitting the integral in (G.1) into 2 integrals, the former over the interval $0 \leq u \leq 1$ which is then developed in series, the latter over the interval $1 \leq u \leq \infty$, which, being uniformly convergent inside $\mathbb{C}$, provides an entire function. The terms of the series (uniformly convergent inside D_Γ) provide the *principal parts* of $\Gamma(z)$ at the corresponding poles $z_n = -n$.

Reflection or Complementary Formula [6]

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,.} \qquad (G.12)$$

This formula, which shows the relationship between the Γ function and the trigonometric function, is of great importance together with the recurrence formula (G.3). It can be proven in several manners; the simplest proof consists in proving (G.12) for $0 < \mathcal{R}e(z) < 1$ and extend the result by analytic continuation to $\mathbb{C}$ except the points $0, \pm 1, \pm 2, \ldots$

The reflection formula shows that $\Gamma(z)$ has no zeros. In fact, the zeros cannot be in $z = 0, \pm 1, \pm 2, \ldots$ and, if $\Gamma(z)$ vanished for a not integer z, because of (G.12) this zero be a pole of $\Gamma(1-z)$, that cannot be true.

Multiplication Formulas [7–8]

Gauss proved the following *Multiplication Formula* [7]

$$\boxed{\Gamma(nz) = (2\pi)^{(1-n)/2}\, n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n})\,, \quad n = 2, 3, \ldots\,,} \qquad (G.13)$$

which reduces, for $n = 2$, to *Legendre's Duplication Formula* [8]

$$\boxed{\Gamma(2z) = \frac{1}{\sqrt{2\pi}}\, 2^{2z-1/2}\, \Gamma(z)\, \Gamma(z + \frac{1}{2})\,,} \qquad (G.14)$$

and, for $n = 3$ to the *Triplication Formula*

$$\boxed{\Gamma(3z) = \frac{1}{2\pi}\, 3^{3z-1/2}\, \Gamma(z)\, \Gamma(z + \frac{1}{3})\, \Gamma(z + \frac{2}{3})\,.} \qquad (G.15)$$

Pochhammer's Symbols [9]

Pochhammer's symbols $(z)_n$ are defined for any non negative integer n as

$$\boxed{(z)_n := z\,(z+1)\,(z+2)\ldots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}\,, \quad n \in \mathbb{N}\,.} \qquad (G.16)$$

with $(z)_0 = 1$. In particular, for $z = 1/2$, we obtain from (G.9)

$$\left(\frac{1}{2}\right)_n := \frac{\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{(2n-1)!!}{2^n}\,.$$

Here, we take the liberty of extending the above notation to negative integers, defining

$$\boxed{(z)_{-n} := z\,(z-1)\,(z-2)\ldots(z-n+1) = \frac{\Gamma(z+1)}{\Gamma(z-n+1)}\,, \quad n \in \mathbb{N}\,.} \qquad (G.17)$$

Graphical Representation of the Gamma Function on the Real Axis [10]

One can have an idea of the graph of the Gamma function on the real axis using the formulas

$$\Gamma(x+1) = x\Gamma(x)\,, \quad \Gamma(x-1) = \frac{\Gamma(x)}{x-1}\,,$$

to be iterated starting from the interval $0 < x \le 1$, where $\Gamma(x) \to +\infty$ as $x \to 0^+$ and $\Gamma(1) = 1$. For $x > 0$ Euler's integral representation (G.1) yields $\Gamma(x) > 0$ and $\Gamma''(x) > 0$ since [10]

$$\Gamma(x) = \int_0^\infty e^{-u}\, u^{x-1}\, du\,, \quad \Gamma''(x) = \int_0^\infty e^{-u}\, u^{x-1}\, (\log\, u)^2\, du\,.$$

As a consequence, on the positive real axis $\Gamma(x)$ turns out to be positive and *convex* so that it is either a monotonic decreasing function, or it first decreases and then increases exhibiting a minimum value. Since $\Gamma(1) = \Gamma(2) = 1$, we must have a minimum at some x_0, $1 < x_0 < 2$. It turns out to be $x_0 = 1.4616\ldots$ where $\Gamma(x_0) = 0.8856\ldots$; hence x_0 is quite close to the point $x = 1.5$ where Γ attains the value $\sqrt{\pi}/2 = 0.8862\ldots$.

On the negative real axis $\Gamma(x)$ exhibits vertical asymptotes at $x = -n$ $(n = 0, 1, 2, \ldots)$; it turns out to be positive for $-2 < x < -1$, $-4 < x < -3$, ..., and negative for $-1 < x < 0$, $-3 < x < -2$, ...

Plots of $\Gamma(x)$ (continuous line) and $1/\Gamma(x)$ (dashed line) are shown for $-4 \le x \le 4$ in Fig. 1, and for $0 \le x \le 3$ in Fig. 2.

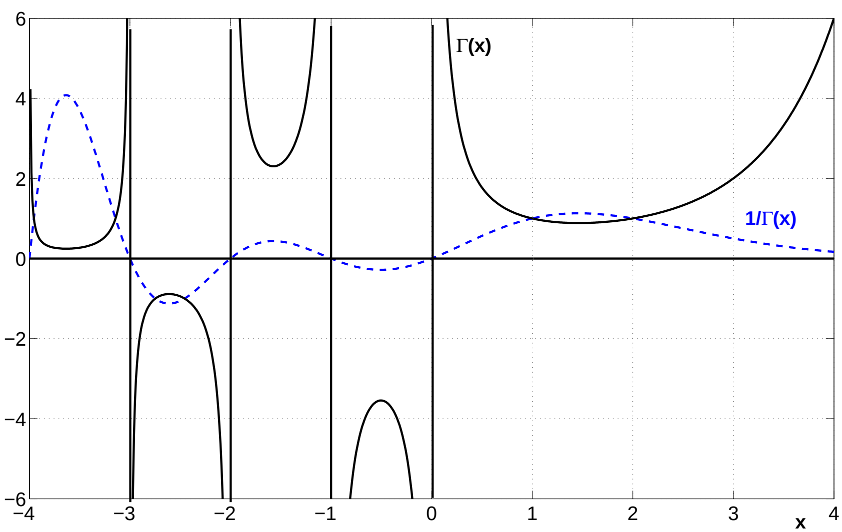

Fig. 1

Plots of $\Gamma(x)$ *(continuous line) and* $1/\Gamma(x)$ *(dashed line) for* $-4 \le x \le 4$

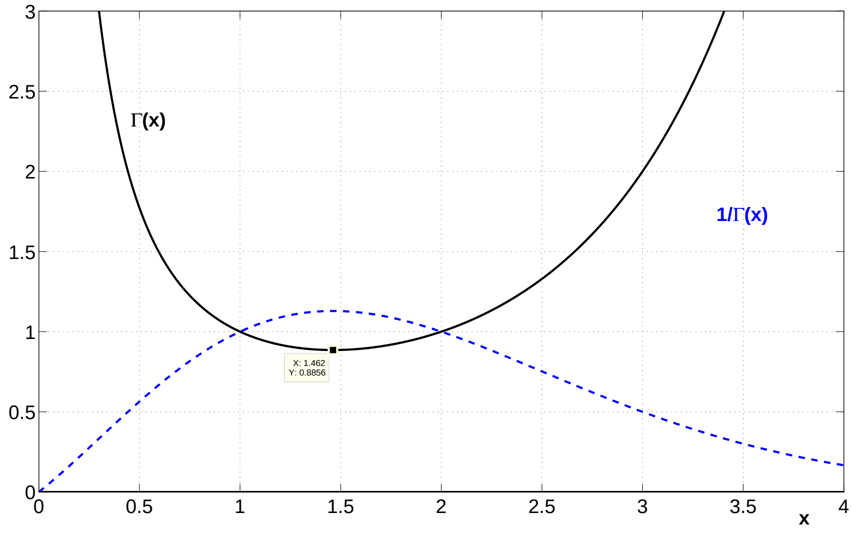

Fig. 2

Plots of $\Gamma(x)$ *(continuous line) and* $1/\Gamma(x)$ *(dashed line) for* $0 < x \le 3$

Analytical Continuation by Hankel's Integral Representations (11–12–13)

Hankel (1864) provided a complex integral representation of the function $1/\Gamma(z)$ valid for *unrestricted* z; it reads (11)

$$\boxed{\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha_-} \frac{e^t}{t^z}\, dt\,, \quad z \in \mathbb{C}\,,} \qquad (G.18a)$$

where Ha_- denotes the Hankel path defined as a contour that begins at $t = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $t = -\infty + ib$ $(b > 0)$. Of course, the branch cut is present when z is not integer because t^{-z} is a multivalued function; in this case the contour can be chosen as in Fig. 3 where

$$\arg(t) = \begin{cases} +\pi\,, & \text{above the cut,} \\ -\pi\,, & \text{below the cut.} \end{cases}$$

When z is an integer, the contour can be taken to be simply a circle around the origin, described in the counterclockwise direction.

An alternative representation is obtained assuming the branch cut along the positive real axis; in this case we get

$$\boxed{\frac{1}{\Gamma(z)} = -\frac{1}{2\pi i} \int_{Ha_+} \frac{e^{-t}}{(-t)^z}\, dt\,, \quad z \in \mathbb{C}\,,} \qquad (G.18b)$$

where Ha_+ denotes the Hankel path defined as a contour that begins at $t = +\infty + ib$ $(b > 0)$, encircles the branch cut that lies along the positive real axis, and ends up at $t = +\infty - ia$ $(a > 0)$. When z is not integer the contour can be chosen as in Fig. 4 where

$$\arg(t) = \begin{cases} 0\,, & \text{above the cut,} \\ 2\pi\,, & \text{below the cut.} \end{cases}$$

When z is an integer, the contour can be taken to be simply a circle around the origin, described in the counterclockwise direction.

We note that $Ha_- \to Ha_+$ if $t \to t\,\mathrm{e}^{-i\pi}$, while $Ha_+ \to Ha_-$ if $t \to t\,\mathrm{e}^{+i\pi}$.

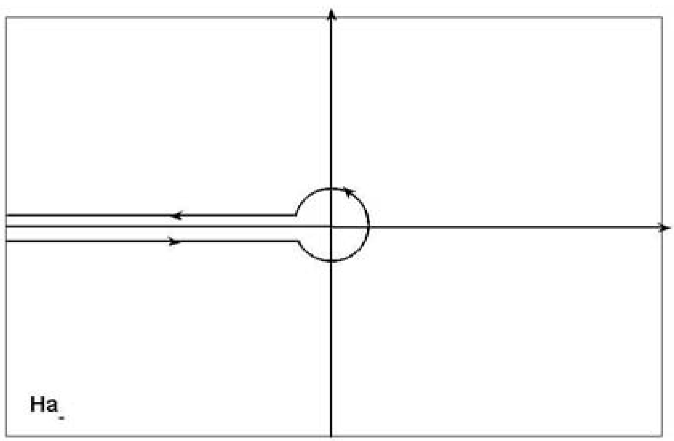

Fig. 3

The left Hankel Contour Ha_-

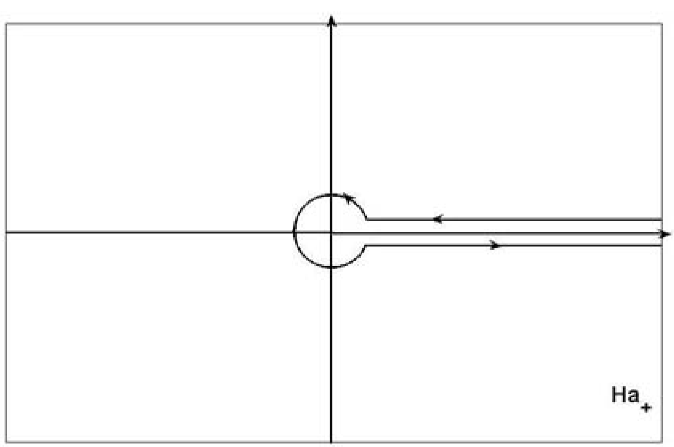

Fig. 4

The right Hankel Contour Ha_-

The advantage of Hankel's representations (G.18) over Euler's integral representation (G.1) is that they converge for *all* complex z and not just for $\mathcal{R}e(z) > 0$. As a consequence $1/\Gamma(z)$ is a transcendental *entire* function (of maximum exponential type); the point at infinity is an essential non isolated singularity, which is an accumulation point of zeros ($z_n = -n\,,\ n = 0, 1, \ldots$). Since $1/\Gamma(z)$ is entire, $\Gamma(z)$ does not vanish in $\mathbb{C}$.

The formulas (G.18) are very useful for deriving integral representations in the complex plane for several *special functions*.

Furthermore, using the reflection formula (G.12), we can get the integral representations of $\Gamma(z)$ itself in terms of the Hankel paths (referred to as *Hankel's Integral Representations* for $\Gamma(z)$), which turn out to be valid in the whole *Domain of Analyticity* D_Γ [12] These representations, which provide the required analytical continuation of $\Gamma(z)$, are

using the path Ha_-

$$\boxed{\Gamma(z) = \frac{1}{2i\,\sin \pi z} \int_{Ha_-} e^{t}\, t^{z-1}\, dt\,, \quad z \in D_\Gamma\,;} \tag{G.19a}$$

using the path Ha_+

$$\boxed{\Gamma(z) = -\frac{1}{2i\,\sin \pi z} \int_{Ha_+} e^{-t}\, (-t)^{z-1}\, dt\,, \quad z \in D_\Gamma\,.} \tag{G.19b}$$

Finall, from (G.19b) we get the formula [13]

$$\boxed{\Gamma(z) = \frac{G(z)}{e^{2\pi i z} - 1}\,, \quad z \in D_\Gamma\,; \quad G(z) := \int_{Ha_+} e^{-t}\, t^{z-1}\, dt} \tag{G.20}$$

where $G(z)$ has the same integrand as Euler's integral representation of $\Gamma(z)$.

Notable Integrals [14–16]

$$\boxed{\int_0^\infty e^{-st}\, t^\alpha\, dt = \frac{\Gamma(\alpha + 1)}{s^{\alpha + 1}}\,, \quad \mathcal{R}e(s) > 0\,, \quad \mathcal{R}e(\alpha) > -1\,.} \tag{G.21}$$

This formula [14] provides the *Laplace transform of the power function* t^α; in particular it shows that the Laplace transform of this function exists for $\mathcal{R}e(\alpha) > -1$ with abscissa of convergence $\sigma_c = 0$.

$$\boxed{\int_0^{\infty} \mathrm{e}^{-at^{\beta}}\, dt = \frac{\Gamma(1+1/\beta)}{a^{1/\beta}}\,, \quad \mathcal{R}e(a) > 0\,, \quad \beta > 0\,.} \qquad (G.22)$$

This formula [(15)] provides for fixed a and $\beta = 2$ the Gauss integral. Therefore, for fixed a, the l.h.s. of (G.22) may be referred to as the *generalized Gauss integral*. Plots of the integral for $a = 1$ versus β are reported in Fig. 5, from which we note that the minimum value is attained at $\beta_0 = 2.16638\ldots$ and holds $I(\beta_0) = 0.8856\ldots$

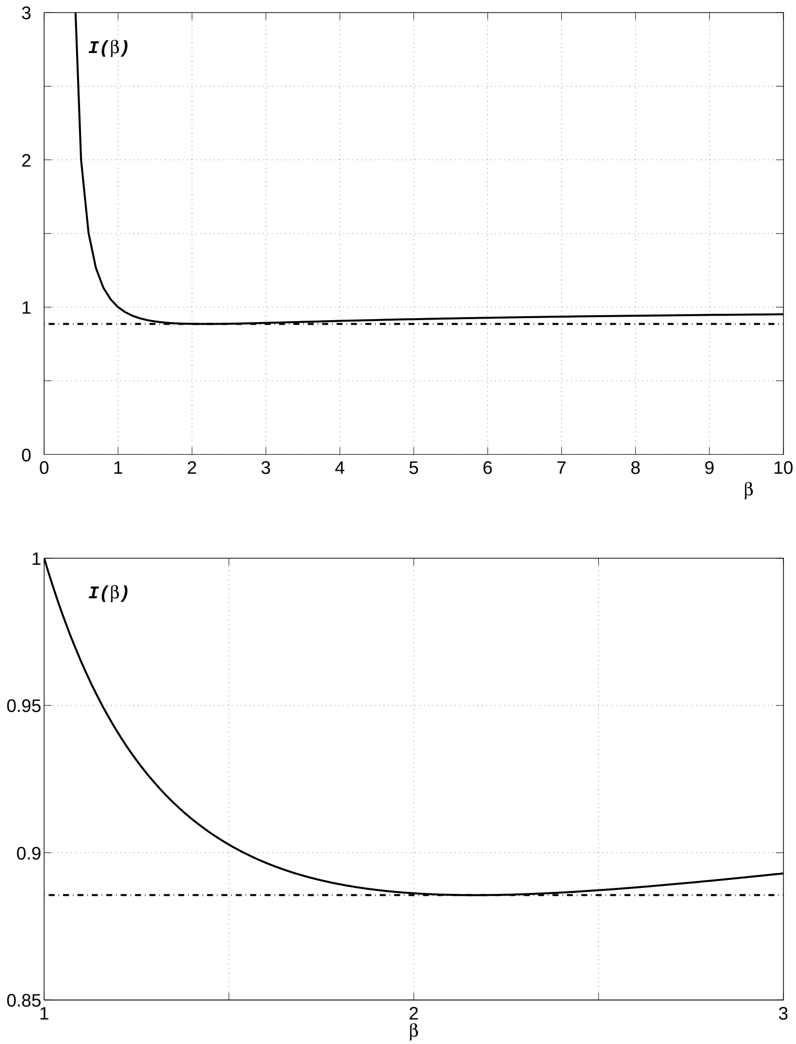

Fig. 5

Plots of the function $I(\beta) = \Gamma(1 + 1/\beta)$ *versus* β,
TOP: for $0 < \beta \leq 10$, *BOTTOM: zoom for* $1 \leq \beta \leq 3$.

$$\boxed{\int_0^\infty \mathrm{e}^{-zt^\mu}\, t^{\nu\,-\,1}\, dt = \frac{1}{\mu}\,\frac{\Gamma(\nu/\mu)}{z^{\nu/\mu}} = \frac{1}{\nu}\,\frac{\Gamma(1+\nu/\mu)}{z^{\nu/\mu}}\,,\ \mathcal{R}e(z)>0\,,\ \mu>0\,,\ \mathcal{R}e(\nu)>0\,.} \qquad (G.23)$$

This formula [16] contains (G.21-22); it reduces to (G.21) for $\{z=s\,,\ \mu=1\,,\ \nu=\alpha+1\}$, and to (G.22) for $\{z=a\,,\ \mu=\beta\,,\ \nu=1\}$.

Asymptotic Formulas

$$\boxed{\Gamma(z) \simeq \sqrt{2\,\pi}\,\mathrm{e}^{-z}\,z^{z-1/2}\left[1+\frac{1}{12\,z}+\frac{1}{288\,z^2}+\ldots\right]\ ; \quad z\to\infty\,, \quad |\arg z|<\pi\,.} \qquad (G.24)$$

This asymptotic expression is usually referred to as *Stirling's Formula*, originally given for $n!$. The accuracy of the formula is surprisingly very good on the positive real axis also for moderate values of $z=x>0$, as it can be noted from the following exact formula,

$$\boxed{x! = \sqrt{2\,\pi}\,\mathrm{e}^{\left(-x+\frac{\theta}{12x}\right)}\,x^{x+1/2}\,; \quad x>0\,, \quad 0<\theta<1\,.} \qquad (G.25)$$

In Fig. 6 we show the comparison between the plot of the Gamma function (continuous line) with that provided by the first term of the Stirling approximation (in dashed line), in the range $0\le x\le 4$

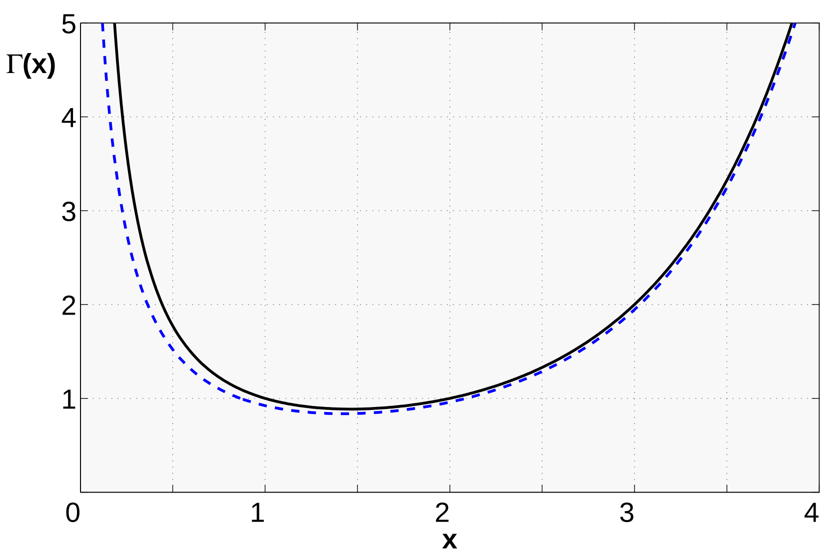

Fig. 6

The $\Gamma(x)$ (continuous line) compared with its first order Stirling aproximation (dashed line)

In Fig. 7 we show the relative error of the first term approximation with respect to the exact value in the range $1 \le x \le 10$; we note that this error decreases from less than $8\,\%$ to less than $1\,\%$.

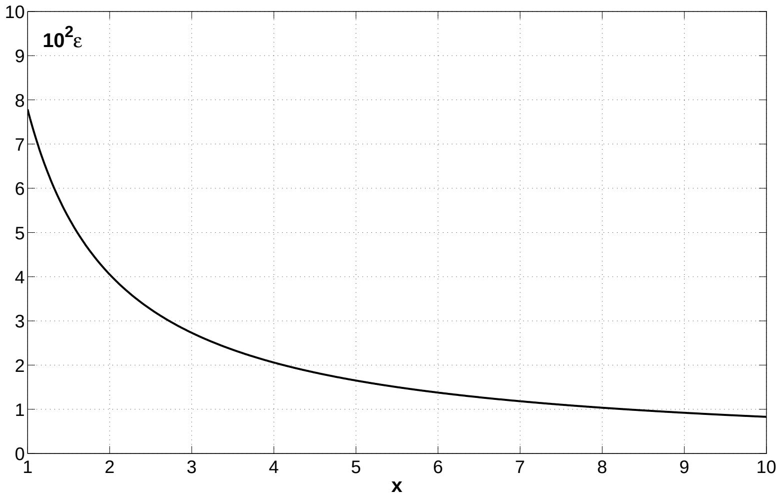

Fig. 7

Plot of the relative error for the first Stirling approximation to $\Gamma(x)$ for $1 \le x \le 10$.

The two following asymptotic expressions provide a generalization of the Stirling formula.

$$\boxed{\Gamma(az+b) \simeq \sqrt{2\,\pi}\,\mathrm{e}^{-az}\,(az)^{az+b-1/2}\,; \quad z \to \infty\,, \quad |\arg z| < \pi\,, \quad a > 0\,.} \tag{G.26}$$

$$\boxed{\frac{\Gamma(z+a)}{\Gamma(z+b)} \simeq z^{a-b}\,\left[1 + \frac{(a-b)\,(a+b-1)}{2z} + \dots\right]\,,} \tag{G.27}$$

as $z \to \infty$ along any curve joining $z = 0$ and $z = \infty$ providing $z \neq -a, -a-1, \dots$, and $z \neq -b, -b-1, \dots$.

THE BETA FUNCTION : $B(p,q)$

Euler's Integral Representation

$$\boxed{B(p,q) = \int_0^1 u^{p-1}\,(1-u)^{q-1}\,du\,, \quad \mathcal{R}e(p) > 0\,, \quad \mathcal{R}e(q) > 0\,.} \tag{B.1}$$

This representation is the standard one for the Beta function, which is also referred to as the *Euler integral of the first kind*, while the integral representation (G.1) for $\Gamma(z)$ is referred to as the *Euler integral of the second kind*. The Beta function is therefore a complex function of two complex variables whose analyticity properties can be deduced later, as soon as the relation with the Gamma function will be established.

Symmetry

$$\boxed{B(p,q) = B(q,p)\,.} \tag{B.2}$$

This property is a simple consequence of the definition (B.1).

Trigonometric Integral Representation

$$\boxed{B(p,q) = 2\int_0^{\pi/2} (\cos\,\vartheta)^{2p-1}\,(\sin\,\vartheta)^{2q-1}\,d\vartheta\,, \quad \mathcal{R}e(p) > 0\,, \quad \mathcal{R}e(q) > 0\,.} \tag{B.3}$$

This noteworthy representation follows from (B.1) by setting $u = (\cos\,\vartheta)^2$.

Relation with the Gamma Function

$$\boxed{B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,.} \tag{B.4}$$

This relation is of fundamental importance. Furthermore, it allows us to obtain the analytical continuation of the Beta function. The proof of (B.4) can be easily obtained by writing the product $\Gamma(p)\,\Gamma(q)$ as a double integral which is to be evaluated introducing polar coordinates. In this respect we must use the Gaussian representation (G.7) for the Gamma function and the trigonometric representation (B.3) for the Beta function. In fact,

$$\begin{aligned}
\Gamma(p)\,\Gamma(q) &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,u^{2p-1}\,v^{2q-1}\,du\,dv \\
&= 4\int_0^\infty e^{-\rho^2}\,\rho^{2(p+q)-1}\,d\rho\int_0^{\pi/2}(\cos\,\vartheta)^{2p-1}\,(\sin\,\vartheta)^{2q-1}\,d\vartheta \\
&= \Gamma(p+q)\,B(p,q).
\end{aligned}$$

Henceforth, we shall exhibit other integral representations for $B(p,q)$, all valid for $\mathcal{R}e(p) > 0$ and $\mathcal{R}e(q) > 0$.

Integral Representations on $[0,\infty)$

$$\boxed{B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\, dx = \int_0^\infty \frac{x^{q-1}}{(1+x)^{p+q}}\, dx = \frac{1}{2}\int_0^\infty \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\, dx\,.} \qquad (B.5)$$

The first representation follows from (B.1) by setting $u = x/(1+x)$; then, the other two are easily obtained by using the symmetry property of $B(p,q)$.

A further Integral Representation on $[0,1]$

$$\boxed{B(p,q) = \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}\, dy\,.} \qquad (B.6)$$

This representation is obtained from the first integral in (B.5) as a sum of two contributions, on $[0,1]$ and $[1,\infty)$.

The Beta function plays a fundamental role in the Laplace convolution of power functions. We recall that the Laplace convolution is the convolution between causal functions (*i.e.* vanishing for $t<0$),

$$f(t) \,*\, g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau\,.$$

The convolution satisfies both the commutative and associative properties in that

$$f(t) \,*\, g(t) = g(t) \,*\, f(t)\,, \quad f(t) \,*\, [g(t) \,*\, h(t)] = [f(t) \,*\, g(t)] \,*\, h(t)\,.$$

It is straightforward to show by setting in (B1) $u = \tau/t$, the

Convolution Representation

$$\boxed{t^{p-1} \,*\, t^{q-1} = \int_0^t \tau^{p-1}\,(t-\tau)^{q-1}\, d\tau = t^{p+q-1}\, B(p,q)\,.} \qquad (B.7)$$

Introducing the causal Gel'fand-Shilov function

$$\Phi_\lambda(t) := \frac{t_+^{\lambda-1}}{\Gamma(\lambda)}\,, \quad \lambda \in \mathbb{C}\,,$$

(where the suffix $+$ just denotes the causality property of vanishing for $t<0$), we can write the previous result in the following interesting form

Convolution between Gel'fand-Shilov Functions

$$\boxed{\Phi_p(t) \,*\, \Phi_q(t) = \Phi_{p+q}(t)\,.} \qquad (B.8)$$

In fact, dividing by $\Gamma(p)\,\Gamma(q)$ the L.H.S of (B.7), and using (B.4), we just obtain (B.8).

Some Applications of the Beta Function

The results (B.7-8) show that the convolution integral between two (causal) functions, which are absolutely integrable in any interval $[0,t]$ and bounded in every finite interval that does not include the origin, is not necessarily continuous at $t = 0$, even if a theorem ensures that this integral turns out to be continuous for any $t > 0$, see G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer-Verlag, Berlin 1974, pp. 47-48. In fact, considering two arbitrary real numbers α, β greater than -1, we have

$$I_{\alpha,\beta}(t) := t^\alpha * t^\beta = B(\alpha{+}1,\beta{+}1)\, t^{\alpha+\beta+1} \Rightarrow \lim_{t\to 0^+} I_{\alpha,\beta}(t) = \begin{cases} +\infty & \text{if } -2 < \alpha+\beta < -1\,, \\ c(\alpha) & \text{if } \quad \alpha+\beta = -1\,, \\ 0 & \text{if } \quad \alpha+\beta > -1\,, \end{cases}$$

where $c(\alpha) = B(\alpha+1, -\alpha) = \Gamma(\alpha+1)\,\Gamma(-\alpha) = \pi/\sin(-\alpha\pi)$.

We note that in the case $\alpha + \beta = -1$ (and therefore $-1 < \alpha < 0$) the convolution integral attains for any $t > 0$ the constant value $c(\alpha) \geq \pi$. In particular, for $\alpha = \beta = -1/2$, we obtain the minimum value for $c(\alpha)$ (relevant result in the problem of the *tautochrone*), *i.e.*

$$\boxed{\int_0^t \frac{d\tau}{\sqrt{\tau}\,\sqrt{t-\tau}} = \pi\,.} \qquad (B.9)$$

The Beta function is also used to prove some basic identities for the Gamma function, like the Complementary Formula (G.12) and the Duplication Formula (G.14).

For the Complementary Formula we know that it is sufficient to prove it for $0 < \alpha < 1$,

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}\,, \quad 0 < \alpha < 1\,.$$

We note from (B.4-5) that

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = B(\alpha, 1-\alpha) = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx\,,$$

and from a classical exercise in complex analysis

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\pi\alpha}\,.} \qquad (B.10)$$

For the Duplication Formula we note that it is equivalent to

$$\Gamma(1/2)\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+1/2)\,,$$

and hence, after simple manipulations, to

$$B(z, 1/2) = 2^{2z-1}\,B(z,z)\,.$$

This identity is easily verified for $\mathcal{R}e(z) > 0$, using the trigonometric representation (B.3) for the Beta function and noting that

$$\int_0^{\pi/2} (\cos\vartheta)^\alpha\,d\vartheta = \int_0^{\pi/2} (\sin\vartheta)^\alpha\,d\vartheta \;= 2^\alpha \int_0^{\pi/2} (\cos\vartheta)^\alpha\,(\sin\vartheta)^\alpha\,d\vartheta\,, \quad \mathcal{R}e(\alpha) > -1\,,$$

since $\sin 2\vartheta = 2\,\sin\vartheta\,\cos\vartheta$.

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