An approach to the *three sisters* functions

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Introduction

In this work I will deal with some basic Laplace Transform pairs, the *3 sisters*, which are helpful in the treatment of the classical (partial differential) *diffusion equation*. I will show they may be obtained by using some important features of the theory of *Laplace Transforms* and the definitions of the *error functions*.

Laplace Transform

The Laplace Transform (or L-Transform or L-T) is one of the *integral transforms*, i.e. correspondences between functions through integrals: e.g. $F(z) = \int K(z, z')f(z')dz'$ (where $z \in \mathbb{C}$ and K is a kernel) associates $f(z')$ with $F(z)$.

For the aim of this work, a complete treatment of Laplace Transform is not needed: I will limit to the definition and to the properties that are relevant to the *3 sisters* and the *error functions*.

Consider a causal and locally summable[1] function $f(t)$, $t \in \mathbb{R}$. The *Laplace Transform* $\mathcal{L}[f(t)]$ of $f(t)$ is defined by the following integral (if it exists)

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)dt \tag{1}$$

where $s \in \mathbb{C}$. The common notation to synthetically point out a function and its Laplace Transform is: $f(t) \div \tilde{f}(s)$. One may demonstrate that, if the integral in eq. (1) is convergent for s_0, then it is convergent for every s whose real part is greater than the real part of s_0; so the Laplace Transform is defined in a half plane. The lower bound of the real parts of s_0 is the abscissa of convergence σ_c. Only if $\Re[s] > \sigma_c$ the integral in eq. (1) is convergent. Besides, we may say that the integral in eq. (1) is *uniformly convergent* if

$$\mathcal{L}[f(t)] = \lim_{T \to +\infty} \int_0^T \mathrm{e}^{-st} f(t)dt \tag{2}$$

is independent from s.

A notable theorem (the *initial and final values theorem*) relates $f(t)$ and its L-Transform

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, \tilde{f}(s) \tag{3}$$

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s\, \tilde{f}(s). \tag{4}$$

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[1] That is $\int_0^T |f(t)| < \infty$.

Evidently when one of the four limits does not exist, the theorem does not hold.

Some helpful L-Transforms for elementary functions together with the abscissa of convergence are listed in tab. 1. In addition Laplace Transform

$f(t)$	$\tilde{f}(s)$	σ_c
1	$\frac{1}{s}$	0
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	α
t^{α} (with $\Re[\alpha] > -1$)	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	0

Table 1: Some important Laplace Transforms. Here we suppose $\alpha \in \mathbb{C}$.

has a number of properties which allow to get L-Tranforms of more complex functions. Here are the most significant ones (let $\sigma_c(\tilde{f}(s)) = \lambda$)

1. **linearity**

$$\sum_{k=1}^{N} c_k f_k(t) \div \sum_{k=1}^{N} c_k \tilde{f}_k(s)\ , \text{ with } \sigma_c = \max\{\lambda_k\} \tag{5}$$

2. **scale change** ($a \in \mathbb{R}^+$)

$$f(at) \div \frac{1}{a}\,\tilde{f}(\frac{s}{a})\ , \text{ with } \sigma_c = a\lambda \tag{6}$$

3. **translation** to the right ($a \in \mathbb{R}^+$ and $\theta(t-a)$ is the Heaviside function[2])

$$f(t-a)\,\theta(t-a) \div e^{-as}\tilde{f}(s)\ , \text{ with } \sigma_c = \lambda \tag{7}$$

4. **translation** to the left ($a \in \mathbb{R}^+$)

$$f(t+a) \div e^{as}\left[\tilde{f}(s) - \int_0^a e^{-st}\, f(t)\, d(t)\right] \tag{8}$$

5. **multiplication by** $e^{\alpha t}$ ($\alpha \in \mathbb{C}$)

$$f(t)\, e^{\alpha t} \div \tilde{f}(s-\alpha)\ , \text{ with } \sigma_c = \lambda + \Re[\alpha] \tag{9}$$

[2]Defined as $\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$.

6. **multiplication by t^n**

$$t^n\, f(t) \div (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\, \tilde{f}(s)\ ,\ \text{with } \sigma_c = \lambda \tag{10}$$

7. **division by t**

$$\frac{f(t)}{t} \div \int_s^\infty \tilde{f}(s)\,\mathrm{d}s \tag{11}$$

8. **derivation**

$$\frac{\mathrm{d}^n f}{\mathrm{d}t^n} \div s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-1-k}\, f^{(k)}(0^+)\ ,\ \text{with } \sigma_c = \max(\sigma_c(\mathcal{L}(\frac{\mathrm{d}f}{\mathrm{d}t})), 0) \tag{12}$$

9. **integration**

$$\int_0^t f(t')\,\mathrm{d}t' \div \frac{\tilde{f}(s)}{s}\ ,\ \text{with } \sigma_c = \max(\lambda, 0) \tag{13}$$

10. **convolution**

$$f * g \div \tilde{f}(s)\, \tilde{g}(s) \tag{14}$$

11. **causal periodic functions**

$$f(t) = f(t + kT) \div \frac{\int_0^T \mathrm{e}^{-st}\, f(t)\,\mathrm{d}t}{1 - \mathrm{e}^{-sT}}. \tag{15}$$

Anti-Laplace Transform

Given $\tilde{f}(s)$, it is possible to go back to $f(t)$: one may demonstrate that (if $\tilde{f}(s)$ is analytical and $\lim_{s\to\infty} \tilde{f}(s) = 0$)

$$f(t) = \frac{1}{2\pi\imath} \int_{\sigma-\imath\infty}^{\sigma+\imath\infty} \mathrm{e}^{st}\, \tilde{f}(s)\,\mathrm{d}s \tag{16}$$

with $\sigma \in \mathbb{R}$, $\sigma > \sigma_c$. The function $f(t)$ must satisfy the same conditions that are valid for eq. 1. When the function $\tilde{f}(s)$ has at least one singularity, to compute the integral in eq. (16) one may use a simple theorem, the *Bromwich formula*, known as *Heaviside formula* too: if the following assumptions are met

- $\tilde{f}(s) \to 0$ for $s \to \infty$

- $\tilde{f}(s)$ is monodrome
- $\tilde{f}(s)$ has a finite number of singularities

then

$$f(t) = \sum_k \mathrm{Res}_k \left(\mathrm{e}^{st}\ \tilde{f}(s)\right) \tag{17}$$

where the sum is made over all finite singularities.

The error functions

The *error function* is an entire function defined by

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-\zeta^2}\, \mathrm{d}\zeta \tag{18}$$

where $z, \zeta \in \mathbb{C}$. The integral is independent of the path. In $z = \infty$ there is an essential singularity because $\nexists \lim_{z\to\infty} \operatorname{erf}(z)$, but if $z \in \mathbb{R}$, then $\lim_{z\to\pm\infty} \operatorname{erf}(z) = \pm 1$.

The *complementary error function* is an entire function defined by

$$\operatorname{erfc}(z) \equiv 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-\zeta^2}\, \mathrm{d}\zeta = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{e}^{-\zeta^2}\, \mathrm{d}\zeta \tag{19}$$

where $z, \zeta \in \mathbb{C}$. The path of integration is subjected to the only restriction that $\arg\zeta \to \theta$ with $|\theta| < \pi/4$ as $\zeta \to \infty$. The extreme of integration could be written as $\mathrm{e}^{\imath\theta}\infty$ and hence in particular $+\infty$ when it is assumed $\theta = 0$.

It is usually defined a related entire function, with $z \in \mathbb{C}$:

$$w(z) := \mathrm{e}^{-z^2}\left(1 + \frac{2\imath}{\sqrt{\pi}} \int_0^z \mathrm{e}^{\eta^2}\, \mathrm{d}\eta\right) = \mathrm{e}^{-z^2}\operatorname{erfc}(-\imath z). \tag{20}$$

From definitions above, some *symmetry relations* may be inferred

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) \tag{21}$$

$$\operatorname{erf}(\bar{z}) = \overline{\operatorname{erf}(\mathrm{z})} \tag{22}$$

$$w(-z) = 2\mathrm{e}^{-z^2} - w(z) \tag{23}$$

$$w(\bar{z}) = \overline{w(-z)} \tag{24}$$

Recalling the series representation of the exponential function, $\operatorname{erf}(z)$ and $w(z)$ can be developed in Taylor series, with $z \in \mathbb{C}$:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1} \tag{25}$$

$$w(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma(n/2+1)}; \tag{26}$$

it is also possible to build the asymptotic expansion of $\mathrm{erfc}(z)$ for $z \to \infty$ and $|\arg z| < \dfrac{3\pi}{4}$

$$\mathrm{erfc}(z) \simeq \frac{1}{\sqrt{\pi}} \frac{\mathrm{e}^{-z^2}}{z} \left(1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{(2m)!}{(2z)^{2m}} \right). \tag{27}$$

The *three sisters*

The *three sisters* are three functions that are defined through their Laplace Transforms

$$\phi(a,t) \div \frac{\mathrm{e}^{-a\sqrt{s}}}{s} = \tilde{\phi}(a,s) \tag{28}$$

$$\psi(a,t) \div \mathrm{e}^{-a\sqrt{s}} = \tilde{\psi}(a,s) \tag{29}$$

$$\chi(a,t) \div \frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}} = \tilde{\chi}(a,s) \tag{30}$$

where $a, t \in \mathbb{R}^+$ and $\Re[s] > 0$. It is easy to demonstrate that each of them can be expressed as a function of one of the 2 other *three sisters* (table 2).

	$\tilde{\phi}$	$\tilde{\psi}$	$\tilde{\chi}$
$\tilde{\phi}$	$\frac{\mathrm{e}^{-a\sqrt{s}}}{s}$	$\frac{\tilde{\psi}}{s}$	$-\frac{1}{s}\frac{\partial\tilde{\chi}}{\partial a}$
$\tilde{\psi}$	$s\,\tilde{\phi}$	$\mathrm{e}^{-a\sqrt{s}}$	$-\frac{\partial\tilde{\chi}}{\partial a}$
$\tilde{\chi}$	$-\frac{\partial\tilde{\phi}}{\partial a}$	$-\frac{2}{a}\frac{\partial\tilde{\psi}}{\partial s}$	$\frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}}$

Table 2: Relations between the *three sisters* in the Laplace domain.

The *three sisters* in the t domain may be all directly calculated, by making use of the *Bromwich formula* (eq. 17), with which we obtain:

$$\tilde{\phi}(a,s) \div \phi(a,t) = 1 - \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\sin(a\sqrt{r})\frac{\mathrm{d}r}{r} \tag{31}$$

$$\tilde{\psi}(a,s) \div \psi(a,t) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\sin(a\sqrt{r})\,\mathrm{d}r \tag{32}$$

$$\tilde{\chi}(a,s) \div \chi(a,t) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\cos(a\sqrt{r})\frac{\mathrm{d}r}{\sqrt{r}}. \tag{33}$$

Then, through the substitution $\rho = \sqrt{r}$, we may arrive at the Gaussian integral and, consequently, at the required explicit expressions of the *three sisters*:

$$\phi(a,t) = \mathrm{erfc}(\frac{a}{2\sqrt{t}}) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{a/2\sqrt{t}} \mathrm{e}^{-u^2}\, \mathrm{d}u \tag{34}$$

$$\psi(a,t) = \frac{a}{2\sqrt{\pi}}\, t^{-3/2}\, \mathrm{e}^{-a^2/4t} \tag{35}$$

$$\chi(a,t) = \frac{1}{\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-a^2/4t}\ , \tag{36}$$

reminding eq. (18) and (19). Alternatively we can compute the *three sisters* in the t domain by using the relations between the *three sisters* in the Laplace domain listed in table 2. But in this case one of the *three sisters* in the t domain must be already known. Assuming to know $\phi(a,t)$ (eq. 34) we can get:

- $\psi(a,t)$ from $\tilde{\psi}(a,s) = s\,\tilde{\phi}(a,s)$: recalling eq. 12 with $n = 1$ we can write

$$s\,\tilde{\phi}(a,s) \div \frac{\partial}{\partial t}\,\phi(a,t) \tag{37}$$

since $\phi(a,0^+) = 0$. Hence[3]:

$$\psi(a,t) = \frac{a}{2\sqrt{\pi}}\, t^{-3/2}\, \mathrm{e}^{-a^2/4t} \tag{39}$$

- $\chi(a,t)$ from $\tilde{\chi}(a,s) = -\frac{\partial}{\partial a}\tilde{\phi}(a,s)$: as a is a parameter, it immediately follows that

$$\chi(a,t) = -\frac{\partial}{\partial a}\,\phi(a,t) = \frac{1}{\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-a^2/4t} \tag{40}$$

In fig. 1 the *three sisters*, with $a = 1$ are plotted.

[3]To derive $\phi(a,t)$ we may take advantage of a theorem contained in Bononcini (1961) which states that, if $f(x,y)$ and its derivative $\partial f/\partial y$ are continuous functions defined for $a \le x \le b$, $c \le y \le d$ and if $\alpha(y)$ and $\beta(y)$ are derivable $\forall y \in [c,d]$ with $a \le \alpha(y) \le b$, $a \le \beta(y) \le b$, then the function $F(y) = \int_{\alpha(y)}^{\beta(y)} f(x,y)\,\mathrm{d}x$ can be derived and it results

$$\frac{\mathrm{d}}{\mathrm{d}y}F(y) = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}x + \beta'(y) f[\beta(y),y] - \alpha'(y) f[\alpha(y),y]. \tag{38}$$

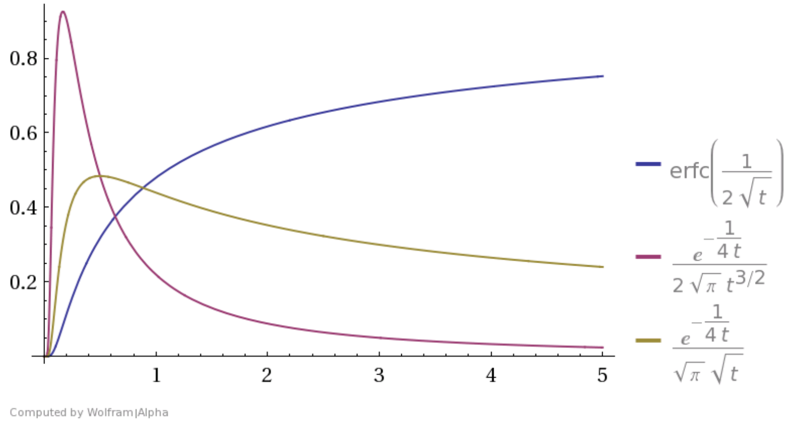

Figure 1: The *three sisters* functions $\phi(a,t)$, $\psi(a,t)$ and $\chi(a,t)$ (with $a=1$) in the t domain.

Another series of Laplace Transform pairs (similar to the *three sisters*) involves the error functions in the Laplace domain

$$\mathrm{e}^{-at^2} \div \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{s^2/4a}\,\mathrm{erfc}\left(\frac{s}{2\sqrt{a}}\right) \quad , \ \forall s \in \mathbb{C} \tag{41}$$

$$\frac{1}{\sqrt{t+a}} \div \sqrt{\frac{\pi}{s}}\,\mathrm{e}^{as}\,\mathrm{erfc}(\sqrt{as}) \ , \ \Re[s] > 0 \tag{42}$$

$$\frac{1}{\sqrt{t}(t+a)} \div \frac{\pi}{\sqrt{a}}\,\mathrm{e}^{as}\,\mathrm{erfc}(\sqrt{as}) \ , \ \Re[s] \geq 0 \tag{43}$$

where $a > 0$.

Bibliography

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